

ACTAS de la ACADEMIA NACIONAL DE CIENCIAS



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ACTAS de la ACADEMIA NACIONAL DE CIENCIAS



TOMO
XVI

CIEM, 40 años de Matemática en Córdoba

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CÓRDOBA - REPÚBLICA ARGENTINA
2024

ACADEMIA NACIONAL DE CIENCIAS. (CÓRDOBA, ARGENTINA)

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PRÓLOGO

En el mes de junio de 2023 se llevó a cabo el congreso "CIEM, 40 años de Matemática en Córdoba", en la Academia Nacional de Ciencias, con el motivo de celebrar, con mucha alegría y orgullo, los primeros cuarenta años de existencia del CIEM, Centro de Investigación y Estudios de Matemática. El presente volumen contiene las actas de dicho congreso.

El CIEM, junto a la FaMAF, son el hogar académico de 140 matemáticos que tuvieron la invaluable oportunidad de educarse en Matemática y formarse como investigadores en Córdoba, al mejor nivel internacional, hoy diseminados por toda la Argentina y varios otros países. Sin lugar a dudas, la Matemática en Córdoba se encuentra actualmente consolidada, pero los desafíos se renuevan. Es muy importante no bajar los brazos y continuar con el desarrollo, la enseñanza y la divulgación de esta bella e imprescindible ciencia que, como dijo Hardy hace ya más de un siglo, "es la más vieja y la más joven de todas".

Quedamos comprometidos todos los matemáticos formados en Córdoba a honrar, en cada clase y en cada artículo, con dedicación y buena calidad, el gran esfuerzo y sacrificio que significó para nuestros Profesores pioneros agregar el nombre de Córdoba en el mapa de la Matemática mundial. Nuestro más profundo agradecimiento hacia ellos.

Estamos muy agradecidos con todos los conferencistas por las charlas y los artículos de excelente calidad, como así también con la Academia Nacional de Ciencias por abrirnos sus puertas para realizar el congreso, es un honor publicar sus trabajos en este volumen XVI de Actas de la Academia Nacional de Ciencias, una serie que data del año 1875.

Jorge Lauret
Académico

Leandro Cagliero

UNA VISIÓN HISTÓRICA DE LA MATEMÁTICA EN CÓRDOBA

JUAN ALFREDO TIRAO

He asumido con mucho agrado la invitación a dirigirme a ustedes en esta celebración del cuadragésimo aniversario de la creación del CIEM. Pienso que es bueno documentar lo conocido y vivido en IMAF y en su continuación FAMAF a lo largo de más de sesenta años. Este legado histórico puede ser valioso para las generaciones más jóvenes, al conocer los antecedentes, dificultades y vicisitudes que jalonaron el camino de una empresa tan importante, cual es implantar la investigación en una universidad y en un país con escaso desarrollo científico. Felizmente podemos constatar que tal empresa ha sido exitosa y que hoy la Universidad Nacional de Córdoba cuenta con un centro matemático de relevancia en el ámbito latinoamericano y reconocido por la comunidad matemática internacional. El comienzo de los estudios superiores en la República Argentina se dio en las últimas décadas del siglo XIX, acompañando un importante avance en la educación y un esbozo de Ciencia en nuestro país.

El Congreso Científico Internacional Americano de 1910, que tuvo lugar en Buenos Aires como parte de las celebraciones del primer centenario, fue una vidriera de lo que la ciencia argentina había logrado hasta el momento y un anuncio de las expectativas sobre lo que se concebía como su brillante futuro. Entre las doce secciones que tuvo el Congreso, la dedicada a la física y matemática contó con el consagrado matemático italiano Vito Volterra y Paul Frank, discípulo de Sophus Lie.

En la Argentina y para la matemática, en el período comprendido entre ambas guerras mundiales se dio el arranque de algo nuevo y diferente. En 1917 llegó Julio Rey Pastor y se encontró con un medio atrasado en siglos frente a la matemática del momento. Poco más de veinte años después

ese mismo medio había producido suficiente matemática nueva como para albergar una institución, la Unión Matemática Argentina fundada en 1936, una publicación la Revista de la UMA creada en 1937 y un centro de investigación, el actual Instituto Beppo Levi que funciona en Rosario desde 1939. Todos estos logros tuvieron que ver con Julio Rey Pastor.

El desarrollo de la Astronomía, la Física y la Matemática, como ciencias puras, se inició en Córdoba con la fundación del Observatorio Astronómico Nacional dispuesta por el Congreso de la Nación a fines de 1868, siendo Presidente Sarmiento y Avellaneda su Ministro de Educación. El Observatorio fue oficialmente inaugurado el 24 de octubre de 1871. Su primer Director fue Benjamín A. Gould.

Las Fuerzas Armadas, por intermedio del Presidente Provisional de la Nación General Pedro Eugenio Aramburu, apoyaron fuertemente el desarrollo de la investigación científica, creando el CONICET, introduciendo la dedicación exclusiva en las universidades y en particular promoviendo la creación del Instituto de Física de Bariloche en 1955, dirigido por José Antonio Balseiro, y del Instituto de Matemática, Astronomía y Física de la Universidad Nacional de Córdoba en 1956, creación propuesta por Enrique Gaviola entonces Director del Observatorio Astronómico.

El 15 de noviembre de 1956, el Rector Interventor Dr. Jorge A. Núñez protocolizó la Ordenanza No. 6/56 por la cual el Consejo Superior de la Universidad Nacional de Córdoba dispuso la creación del IMAF, el que dependería directamente del Rectorado de la Universidad.

El 27 de noviembre Gaviola se dirigió nuevamente al Rector sugiriendo las medidas necesarias para poder iniciar las clases en la fecha estipulada,

1º de marzo de 1957, entre ellas: designación del Director del Instituto, divulgación de su creación y aceptación de donaciones. En esta nota apareció por primera vez la sigla IMAF.

Finalmente el 30 de noviembre, se reunieron, como Comisión Organizadora, el Rector Jorge Núñez, el Decano de la Facultad de Ciencias Exactas, Físicas y Naturales y el Director del Observatorio Astronómico Dr. Enrique Gaviola. En esta reunión procedieron a designar al Dr. Gaviola como Director ad-honorem del Instituto y al Dr. Fernando R. Colomb como Secretario interino, también ad-honorem. Asimismo aprobaron el plan de estudios del primer semestre y encargaron a Gaviola la redacción del plan de estudios definitivo, el cual debía ponerse a consideración de la Comisión.

El 15 de enero de 1957 Gaviola elevó el proyecto respectivo, el que fue aprobado por el Consejo Superior de la Universidad el 26 de febrero. Gaviola se había doctorado en Física en Alemania y varias de sus creencias firmemente arraigadas se reflejan en varias peculiaridades del plan de estudio.

El plan de estudios estaba elaborado presuponiendo la existencia de profesores, todos con dedicación exclusiva que fuesen capaces de dictar varios cursos sucesivamente. Establecía también que un alumno aplazado dos veces en la misma materia quedaba separado del Instituto.

El listado de materias de dicho plan revela que Gaviola no tuvo en cuenta la formación de matemáticos. Reproducimos a continuación el ciclo propedéutico de cuatro semestres de duración, comunes a las tres carreras. El número entre paréntesis que sigue a una materia indica el peso de la misma en puntos para la promoción.

Primer semestre: Cálculo diferencial e integral I (5); Trigonometría y álgebra (3); Física experimental I (5); Taller (carpintería) (5); Química inorgánica I (2); Geometría descriptiva I (dibujo) (2); Idiomas (inglés y alemán obligatorios, ruso e italiano optativos) (6+3).

Segundo semestre: Cálculo diferencial e integral II (5); Geometría proyectiva (5); Física experimental II (5); Trabajos prácticos en Física I (5); Ta-

ller (carpintería) (5); Química inorgánica II (2); Geometría descriptiva II (2); Idiomas (inglés y alemán obligatorios; francés, ruso e italiano optativos) (6+3).

Con el apoyo total del personal del Observatorio y el empuje de Gaviola, en febrero de 1957 se realizó el primer Curso de Ingreso al IMAF. En este curso preparatorio se inscribieron 72 alumnos y lo aprobaron 26. Gaviola solicitó se acordara beca a 22 de ellos.

Como dijimos, los cursos regulares debían comenzar el 1º de marzo de ese año. Pero, sorpresivamente y sin consultar a los otros miembros de la Comisión Organizadora, Gaviola suspendió la iniciación de las clases. Adujo para eso que en el presupuesto vigente no se habían asignado los fondos necesarios para esas 22 becas.

Casi simultáneamente Gaviola implantó un doble horario a los técnicos y auxiliares del Observatorio Astronómico, desdoblado en mañana y tarde, de manera de tener el mismo personal a mano todo el día. Esto fue resistido por el personal afectado y el asunto tuvo que ser tratado por el Consejo Superior de la Universidad.

Se trató este tema en la reunión ordinaria del martes 12 de marzo junto con el de la suspensión de la iniciación de los cursos del IMAF. Sometidos ambos temas a votación se decidió, por un lado, encargar a la Comisión Organizadora y Directiva del IMAF “realizar las gestiones necesarias para que la actividad docente se inicie a la brevedad” y, por otro lado, se aconsejó al Director del Observatorio el establecimiento de horarios en turnos y horas continuas, ya fuese por la mañana o por la tarde.

La reacción de Gaviola fue pedir al Ministro de Educación de la Nación que el Observatorio volviera a depender del Ministerio como había sido durante 83 años.

Enterado el Rector Núñez de este pedido de separación efectuado por Gaviola, de inmediato, el martes 19, dio por terminadas sus funciones como Director del Observatorio Astronómico y Director Interino del IMAF. A últimas horas de la tarde

el Consejo Superior aprobó unánimemente la actitud del Dr. Núñez y la designación del Dr. Livio Grattón como Director del Observatorio y Director Interino del IMAF, a partir del día 20 de marzo de 1957.

El 15 de abril de 1957 se realizó la ceremonia oficial de inauguración del año lectivo en el Aula Magna de la Facultad de Ciencias Exactas, Físicas y Naturales.

Las dificultades por las que atravesó el Instituto en sus primeros años, en cuanto a su personal, quedaron señaladas en la rápida sucesión de directores: a L. Grattón (20/03/57-31/07/57) siguieron Ricardo Pablo Platzek (físico, 01/08/57-31/08/59), Jorge Landi Dessi (astrónomo, 01/09/59-28/02/60), Emilio Machado (matemático, 01/03/60-31/01/62). En estos años predominaron como docentes tanto astrónomos, como físicos e ingenieros.

A mediados de 1961, año en el cual no se sabía qué iba a ocurrir con la existencia del IMAF, algunos proponían cerrarlo, el Consejo Superior autorizó la contratación del Doctor Alberto Pascual Maiztegui, físico, quien fuera nombrado Director después de Machado (01/02/1962-12/06/1973).

Con la llegada de Maiztegui comenzó la consolidación del IMAF. Con la colaboración de profesores visitantes y con el envío de estudiantes de los cursos superiores a cursar sus últimas materias en otros centros de estudio, como las Universidades de Buenos Aires y La Plata y el hoy Instituto Balseiro de Bariloche, se graduó la primera promoción.

La primera Colación de Grados se realizó, significativamente, en el Salón de Grados de la Universidad el 7 de julio de 1962. En ese acto se entregaron seis diplomas; todos fueron de Licenciados en Física. Recién en diciembre de 1964 terminaron su carrera los primeros Licenciados en Matemática (ocho) y también los primeros Licenciados en Astronomía (tres).

La Colación de Grados de diciembre de 1964 se llevó a cabo en el Salón de Actos del Pabellón Argentina y fue presidido por el Presidente de la Nación Dr. Arturo Humberto Illia y el Rector de la

Universidad Dr. Jorge Orgaz. Los primeros Licenciados en Matemática fueron: Gorki Jover, Salvador Daniel Gigena, Lydia Graciela Prieri, Humberto Raúl Alagia, Juan Carlos Amblard, Aroldo Gustavo Kaplan, Francisco Alberto Grünbaum y Juan Alfredo Tirao.

Bajo los auspicios de la UNESCO y la Facultad de Ciencias Exactas y Naturales de la Universidad de Buenos Aires, el profesor Jean Dieudonné de la Universidad de París dictó durante los meses de julio a setiembre de 1962 un curso en Buenos Aires sobre Representaciones de Grupos Compactos y Funciones Esféricas. También dirigió un seminario sobre Álgebra Lineal.

En ese seminario de Álgebra Lineal participaron varios ingenieros, profesores del Departamento de Matemática de la Facultad de Ciencias Exactas, Físicas y Naturales de nuestra universidad. Entre ellos Nicolás Bello, Pedro Luis Cecchi, Dori Dragone, Pier Baldaccini. Ellos tuvieron la genial idea de invitar al profesor Dieudonné a visitar Córdoba. La visita se concretó a través de la Academia Nacional de Ciencias.

Dieudonné estuvo en Córdoba dos o tres días en el mes de julio de 1962 y pronunció dos conferencias en el IMAF. Una de ellas estuvo referida a demostrar que son más numerosas las funciones continuas en un intervalo cerrado de la recta real sin derivadas en todo punto que las que son derivables en algún punto. En la otra consideró diversos resultados de convergencia de las series de Fourier. Los resultados eran clásicos pero formulados en términos de la matemática moderna.

El profesor Dieudonné integraba el grupo de matemáticos franceses conocido como Nicolás Bourbaki, y era poseedor de una vastísima cultura matemática acorde con su estatura y su fuerte personalidad. Deslumbró a la audiencia y en particular atrajo al grupo de estudiantes que estaban cursando su tercer año: Graciela Prieri, María Teresa Vazquez (a quien recordamos con mucho cariño), Humberto Alagia, Juan Carlos Amblard, Ronny Kaplan, Tito Grünbaum y Alfredo Tirao.

Este grupo estuvo a su lado escuchándolo hablar de matemática con mucha atención durante tantas horas como fue posible. A partir de entonces hubo un intercambio de cartas donde Dieudonné indicaba qué era lo más conveniente estudiar para completar la licenciatura en matemática y luego continuar con un doctorado.

En carta fechada en París el 10 de noviembre de 1962 Dieudonné se dirigía a “Gentlemen” y comenzaba diciendo “I have received your letter and I certainly would like to help you as much as I can, although it is certainly difficult to discuss such matters by correspondence, and furthermore, as I have not been teaching regular classes for some years by now, I may be a little out of touch with what may be expected of a student with your present background. What I will tell you is therefore more something like an ideal program of studies, and I cannot guarantee that even with the best determination and serious work, you can actually put it through in the remaining two years of your studies.

Regarding the background you now have, you certainly have acquired serious foundations in Algebra (modern, of course), General Topology and Functions of one Complex Variable if you have mastered the contents of the books you list, which are all excellent. The other books I do not know; whatever the merits of the Castelnuovo book, you have obviously lost your time studying old-fashioned “Analytic Geometry”, a theory which really never existed: it is merely a bad way of doing linear algebra. Of course, I assume that you have not only read these books but also worked through a substantial number of exercises: doing problems in mathematics is the only and unique way of becoming a mathematician, otherwise you may have read all the books in the world and know no better!!”

A continuación decía que debíamos dedicar los dos años restantes de nuestra carrera a cuatro tópicos centrales, que constituyen la base para cualquier estudio que conduzca a hacer investigación moderna en matemática: Análisis Funcional, Grupos de Lie y sus Aplicaciones, Álgebra y Topología

Algebraica. En cada caso recomendaba los textos y hacía comentarios sustanciales sobre los contenidos y forma de abordarlos.

Con el plan de estudios propuesto por Dieudonné esa generación de estudiantes de 1964 le propuso al Director del IMAF lo que querían estudiar. Fue una fortuna que el director fuera Alberto Maiztegui quién después de salir de su sorpresa hizo todo lo posible para que se pudiera llevar a cabo dicho plan.

En los primeros años en el cuerpo de profesores del IMAF predominaron astrónomos, ingenieros y físicos. El primero con formación de matemático, y por varios años el único, fue Emilio A. M. Machado. Más adelante le sucedieron José Norberto Aguirre, Moisés Sprevak, Carmen Casas de Kalnay, Carlos Loiseau, Magdalena Moujan Otaño, Pedro Luis Cecchi y Arcadio Niell (los dos últimos ingenieros por su título).

Dos matemáticos, profesores visitantes durante varios años, ejercieron fuerte influencia en el IMAF, Eduardo H. Zarantonello y Enzo R. Gentile. Sus cursos de grado y posgrado y sus trabajos con los jóvenes graduados dieron un sello a parte de las actividades de investigación del Instituto.

Debe destacarse la colaboración prestada por la Facultad de Ciencias Exactas y Naturales de la Universidad de Buenos Aires, que autorizó a miembros de su personal con dedicación exclusiva a dictar cursos, colaboración que fue disminuyendo pero que aún se mantuvo importante durante casi toda la década de los setenta.

El Plan de Estudios 1961 de la Licenciatura en Matemática fue reemplazado por el Plan 1965 que incorporó las recomendaciones de Dieudonné y cuya implementación fue posible por la eficaz y decidida gestión de Maiztegui que logró la incorporación de nuevos profesores y la contratación de muy calificados profesores visitantes.

Lamentablemente este promisorio período de crecimiento fue interrumpido por el golpe militar del 27 de junio de 1966, que derrocó al presidente Dr. Arturo Humberto Illia y designó como Presidente Provisional al General Juan Carlos Onganía. La Revolución Argentina interrumpió uno de los

períodos más prolíficos y dinámicos de la historia de las universidades argentinas.

Los sucesos acaecidos hicieron trastabillar nuevamente la estabilidad del IMAF, poco afianzada todavía.

En los últimos meses de 1966 y primeros de 1967 varios egresados de IMAF partieron a hacer sus doctorados en universidades norteamericanas: Francisco Alberto Grünbaum en Rockefeller University (Nueva York), Graciela Prieri University of Wisconsin (Madison), Humberto Alagia y Aroldo Kaplan en Washington University (Saint Louis) y Juan Alfredo Tirao e Ignacio Unsain en la Universidad de California (Berkeley). En 1968 Cristián Urbano Sánchez inició sus estudios doctorales en la Universidad de Rutgers (New Jersey).

Después del golpe de Onganía y hasta 1970 el IMAF tuvo serias dificultades para mantener en pie las licenciaturas. El programa de estudios establecido en 1965 para la licenciatura en matemática se vio seriamente afectado. Sólo la abnegada labor de los docentes que se quedaron y la contribución de los profesores visitantes hizo posible la supervivencia de la licenciatura.

En 1970 se entregaron diplomas de licenciados en matemática a: María Cristina Trevisán, Julio Alfredo Méndez, María Josefina Druetta, Marta Norma Parnisari, Isabel Graciela Dotti, Roberto Jorge Miatello y César Jorge Cirelli. En junio se designó Profesor Adjunto a Francisco Alberto Grünbaum, aprovechando su estada hasta fin de año en Córdoba, y en agosto del mismo año se lo designó con igual categoría a Juan Alfredo Tirao. Ambos después de tres años y medio en el exterior y habiendo completado sus doctorados.

Se preparó un nuevo plan de estudios, volviéndose en lo sustancial al plan de 1965 que fue elaborado con los lineamientos dados por Jean Dieudonné. El mismo se puso en vigencia a partir de 1971 con la anuencia del Doctor Alberto Maiztegui quien continuaba como director del IMAF. Éste continúa vigente con algunas modificaciones.

En los primeros años de la década se dictaron los siguientes cursos para graduados: Ecuaciones diferenciales en derivadas parciales por Francisco

A. Grünbaum (1970); Representaciones de grupos de Lie compactos, Variedades diferenciables y Grupos de Lie por Juan A. Tirao (1971); Operadores diferenciales y Teoría espectral por Juan Carlos Amblard (1971); Ecuaciones diferenciales de la física por Arcadio Niell (1971); Topología algebraica por Jorge Edelman (1971); Análisis armónico por Carlos Segovia (1971).

La investigación en el IMAF fue organizada por el Dr. Maiztegui en Grupos de Trabajo, siguiendo el modelo de la física que trajo del Instituto Balseiro. En junio de 1965 en matemática había dos grupos: el de Matemática Aplicada a cargo de Arcadio Niell y el de Matemática Pura a cargo de Carlos Loiseau.

En 1972 se formalizaron los siguientes dos Grupos de Trabajo:

- Análisis Numérico y Computación a cargo de Arcadio Niell e integrado por: Dr. Sady Maurin, y los licenciados Jorge Enrique Smith, Gorki Jover, Marta Norma Parnisari, Oscar Humberto Bustos y Juan Carlos Amblard.
- Geometría y Teoría de Lie siendo responsable del mismo Juan A. Tirao e integrado por los doctores Cristián Urbano Sánchez e Ignacio Unsain, y por los licenciados: León Roque Sinay, Oscar Antonio Cámpoli, Isabel Graciela Dotti, Roberto Jorge Miatello, María Josefina Druetta, José Raúl Martínez, Graciela Birman, Ana María Redolfi y Roberto Oscar Gandulfo. Poco tiempo después se incorporaron, Susana Cándida Fornari, Sofía Rosalía Paczka y Jorge Antonio Vargas.

Los licenciados Redolfi y Gandulfo estaban haciendo sus doctorados en la Universidad de Washington, Saint Louis, Dotti, Miatello y Cámpoli eran ya estudiantes graduados de la Universidad de Rutgers, New Brunswick, y Sinay estaba cursando su doctorado en el Courant Institute de Nueva York. En 1973 Jorge Vargas se incorporó a la Universidad de Columbia para proseguir su doctorado.

Claramente se apreciaba que a partir de 1966 se inició una política de envío de jóvenes licenciados a realizar sus estudios de doctorado en centros del exterior, cuyo regreso al IMAF se produjo a partir del año 1970 (un total de 11 matemáticos entre 1970 y 1985). Este regreso marcó el inicio de las actividades de investigación en matemática.

Entre el 2 y el 21 de agosto de 1971 se desarrolló en Río Ceballos el Primer Seminario Nacional de Álgebra. A instancia del Dr. Maiztegui el mismo fue organizado por Enzo Gentile, Darío Picco y Juan Tirao. Actuaron como directores de cursos los doctores Horacio O'Brien, Juan José Martínez, Carlos Sánchez y Darío Picco. En este seminario participaron dos estudiantes peruanos.

El Segundo Seminario Nacional de Matemática se realizó en Vaquerías entre el 24 de julio y el 17 de agosto de 1972, y fue organizado por Enzo Gentile y Juan Tirao. Se realizó en el Hotel de la Dirección de Asistencia Social del Personal Universitario (DASPU), recientemente adquirido y remodelado y fue el primer congreso que se llevó a cabo en dicho hotel. Entre otros concurrieron a él: Luis Santaló, Alberto Calderón, Alberto González Domínguez, Elizabeth Hornix, Artibano Micali, Orlando Villamayor, Darío Picco, María Luisa Gastaminza y María Inés Platzeck.

Hasta 1994 se realizaron 11 Seminarios Nacionales bianuales, los primeros en el país cuyo objetivo primordial era despertar las vocaciones de los jóvenes universitarios por la investigación matemática.

En el marco de la nueva Ley Orgánica de las Universidades Argentinas, la 17.245 sancionada el 21 de abril de 1967 por Juan Carlos Onganía, el Rector de la Universidad Nacional de Córdoba Ingeniero Rogelio Nores Martínez, quien detentaba también las atribuciones de Consejo Superior, reglamentó las pruebas de ingreso y dispuso tomarla los días 5 y 6 de febrero de 1970. Ésta prueba de ingreso provocó fuertes enfrentamientos entre el estudiantado y las autoridades universitarias.

En ese tiempo la guerrilla organizaba campos subversivos en Tucumán y diversas acciones armadas ocurrían en Córdoba y en todo el país. El viernes 29 de mayo secuestraron en Buenos Aires al

Teniente General Pedro Eugenio Aramburu. El 2 de junio Onganía, por decreto, impuso la pena de muerte. Todo ello motivó su caída y la Junta de Comandantes en Jefe (Alejandro Agustín Lanusse, Pedro Gnavi y Carlos A. Rey) reasumió el mando el 9 de junio de 1970.

Después de un llamado a elecciones con proscripciones, cuando en 1973 asumió la Presidencia de la Nación Héctor J. Cámpora, el Dr. Maiztegui fue obligado a renunciar y se hizo cargo de la dirección del IMAF el Dr. Juan Carlos Gallardo. Las nuevas autoridades pretendieron llevar adelante una serie de medidas, que por insensatas e irrealizables no prosperaron más allá de la agitación negativa que generaron.

Cuando María Estela Martínez de Perón asumió la Presidencia en 1974 su gobierno trató de eliminar el activismo político y forzó cambios sustanciales en los Consejos que gobernaban las universidades. Esto empeoró cuando los militares fueron instruidos por la Presidente de aniquilar a la guerrilla. En ese momento, el IMAF fue intervenido y pasó a depender de la Facultad de Ciencias Exactas, Físicas y Naturales, nombrándose Director del IMAF al Secretario Académico de la Facultad. Repentinamente y por orden superior proveniente de la Presidencia de la Nación el Rector dispuso el reemplazo del director por el Ing. Juan Carlos Cervi. A nuestro juicio esta designación se constituyó en la más nefasta de la historia del IMAF.

En 1976 una nueva dictadura trató de parar los reclamos sociales con una terrible represión política. Las Fuerzas Armadas anularon toda oposición pero a un altísimo costo. El 21 % de los muertos y desaparecidos fueron estudiantes. Para eliminar el activismo político en las universidades cambiaron el gobierno democrático y colegiado poniendo límites estrictos en la admisión de estudiantes e implementaron un moderado arancel universitario. Como ocurrió en 1966, pero en una escala mucho mayor, miles de profesores y estudiantes fueron perseguidos y forzados a emigrar. La intervención militar en la Universidad Nacional de Córdoba se extendió hasta principios de marzo de 1977. El 2 de marzo de 1977 asumió el rectorado el Dr. Jorge A.

Clariá Olmedo. El nuevo Rector invitó al claustro de profesores del IMAF a proponer a un hombre de la casa para designarlo como Director. El claustro reunido decidió proponer la designación de Juan A. Tirao, que se hallaba instalado como profesor en la Universidad de Brasilia. Éste estuvo al frente del Instituto desde abril de 1977 hasta diciembre de 1983. Durante este período se consolidaron las licenciaturas en matemática y física y se estructuraron los doctorados. En el caso de matemática se estableció como obligatorio para obtener el título de Doctor en Matemática, aprobar tres cursos de posgrado, exponer en dos seminarios, aprobar dentro de los dos primeros años un Examen de Matemática Básica (examen de calificación) y aprobar una Tesis Doctoral, régimen aún vigente con algunos cambios. Se consiguió que la universidad asignara un predio para el IMAF dentro de la ciudad universitaria y que se levantara la estructura del primer edificio propio del Instituto. Hoy esta estructura es la del edificio central de unos 5.500 metros cubiertos que ocupa FaMAF y que fue inaugurado en 1993. El IMAF era concebido como un centro destinado a la formación de matemáticos, físicos y astrónomos. Esta función determina una muy específica conjunción de las tareas docentes y de investigación, que no pueden separarse a nivel institucional sin desvirtuar su función. Formar profesionales en estas disciplinas requiere un proceso de educación en un ambiente en el cual la investigación, además de su importancia intrínseca, sea parte esencial de la actividad docente. Estas características definen a una institución como el IMAF, y fijan su posición dentro de la Universidad. Con motivo de la celebración de los XXV años del IMAF el 15 de noviembre de 1981, se organizó un “Ciclo de Conferencias XXV Aniversario” y las “Jornadas Nacionales sobre Formación de Recursos Humanos e Investigación en Matemática, Astronomía y Física”. La dirección contó para su organización con la inestimable colaboración de los Doctores Humberto R. Alagia (Jefe de la Sección Matemática) y Raúl T. Mainardi (Jefe de la Sección Física). Las seis conferencias del Ciclo fueron desarrolladas a lo

largo del año y estuvieron a cargo de los siguientes profesores:

- Dr. Laverne S. Birks (Naval Research Laboratory – EEUU),
- Dr. Ludger Kamp (Universidad de Constanza – Alemania),
- Dr. Enrique Lami Dozo (Universidad Libre de Bruselas – Bélgica),
- Dra. Laura Levi (Instituto de Física de la Atmósfera del Servicio Meteorológico Nacional),
- Dr. Eduardo Zarantonello (IMAF – CONICET),
- Dr. Mario Garavaglia (Centro de Investigaciones Ópticas de La Plata).

Las Jornadas tuvieron lugar el 12 y 13 de noviembre. Después del acto inaugural, donde se entregaron medallas recordatorias a los pocos docentes y no docentes que habían cumplido veinticinco años sirviendo a la institución, se desarrollaron conferencias sobre la problemática de las jornadas a cargo de los siguientes invitados especiales:

- Ing. Juan Rogelio Rodríguez (Subsecretario de Ciencia y Tecnología del Ministerio de Cultura y Educación de la Nación),
- Dr. Mario Garavaglia (Centro de Investigaciones Ópticas de La Plata),
- Dr. Oscar Bressan (Vice Director del Centro Atómico Bariloche),
- Dr. José Luis Sérsic (Observatorio Astronómico de la UNC),
- Dr. Luis Santaló (Presidente de la Academia Nacional de Ciencias Exactas, Físicas y Naturales).

Las jornadas se cerraron con un panel integrado por los conferencistas y los Doctores Orlando Villamayor y Eduardo H. Zarantonello. Actuó como moderador el Profesor Dr. Andrés Raggio. Los panelistas tuvieron la oportunidad, al contestar las preguntas de los participantes, de ampliar sus puntos de vista o de precisar sus ponencias.

Al anochecer del 12 de noviembre se ofreció una velada musical en la sala de conciertos de la Escue-

la de Música de la Universidad. La misma estuvo a cargo de los consagrados pianistas Humberto Catania y Dante Medina, profesores de la Escuela. La celebración se cerró con un copetín servido en una de las casonas próximas a la entrada de la ciudad universitaria.

En 1980 los doctores Humberto Alagia, entonces Jefe de la Sección Matemática del IMAF, y Juan Tirao prepararon toda la documentación necesaria para solicitar la creación de un Programa del CONICET llamado PROMIMAF (Programa de Acción para Actividades de Posgrado e Investigación Matemática del IMAF), establecido el 1/7/1980. De esta forma se contó con una fuente de financiación adicional para la compra de revistas y libros, para cubrir los gastos de pasajes y estadías de profesores invitados, compra de muebles y útiles de librería.

Tres años después (10/05/1983), este programa se transformó en el CIEM (Centro de Investigación y Estudios de Matemática de Córdoba), un instituto de doble dependencia Universidad-CONICET con los mismos objetivos de afianzar el desarrollo matemático dentro del IMAF. El Director del IMAF les ofreció a los físicos y los instó a hacer lo propio y gestionar con su apoyo un instituto de física en el ámbito del IMAF. El primer director del CIEM fue Juan A. Tirao (1983-1995) quien fue sucedido por Cristián U. Sánchez (1995-2011), Roberto J. Miatello (2011-2017), Nicolás Andruskiewitsch (2017-2022) y desde entonces por Jorge Lauret. Fueron secretarías: Ana María Ferreiro (1983-1992), Nancy Moyano desde 1992, María Laura Buyulaqui (1998-2002), María Teresa Bertorello (2009-2010) y Claudia Aguirre desde 2010. Transcurridos cuarenta años desde su creación, ha quedado palmariamente demostrado que el CIEM ha contribuido significativamente al desarrollo de un centro de formación y de investigación de primer nivel. Veinticinco años más tarde se creó el Instituto de Física Enrique Gaviola dentro de FaMAF. El IMAF se había convertido en una unidad académica madura a la que le quedaba chica la estructura de instituto. Durante el segundo semestre de 1983 se gestionó su transformación en facultad

lo cual se logró el 21 de noviembre de 1983 por Decreto Ley 3038/83 de la Presidencia de la Nación. Los que propiciamos esta transformación lo hicimos convencidos de que esto favorecería un mayor desarrollo de la docencia e investigación en matemática, física y astronomía y haría un aporte significativo a la vida académica de la Universidad participando de las decisiones de gobierno del Consejo Superior. Afortunadamente el 10 de diciembre de 1983 la Argentina retomó la democracia y desde entonces las universidades volvieron a ser modeladas sobre los principios de la Reforma Universitaria, basados en el gobierno tripartito. Le tocó al Dr. Humberto Alagia, como Decano Normalizador, conducir la FaMAF en sus primeros pasos como facultad organizando la nueva estructura de gobierno y elaborando toda la normativa involucrada. Desde la creación del doctorado a la actualidad la Facultad tuvo un sostenido crecimiento. Los primeros doctores fueron Alicia García (Oct. 85) y Linda Saal (Dic. 1985), dirigidas por los Dres Cristián Sánchez y Jorge Vargas respectivamente. Ya en el año 2000 la planta de FaMAF contaba con unos 40 doctores en Matemática, la gran mayoría recibidos en FaMAF, más 3 con título de Magíster, observándose ya la presencia de doctores de la segunda generación dirigiendo tesis doctorales.

Una política que contribuyó significativamente al progreso fue el envío desde 1988 de jóvenes investigadores, por medio de becas externas del Conicet, a hacer un posdoctorado en centros del primer nivel en Europa y USA (Ecole Polytechnique, Yale, MIT, University of Düsseldorf, University of Princeton, University of Bonn, University of Berlin, New York University). Esto permitió contar con investigadores ‘aggiornados’ que trajeron a su regreso al país temas de investigación de la mayor actualidad. De este modo llegaron a Córdoba los grupos cuánticos, los usos de la holonomía normal en la geometría de subvariedades, las investigaciones en cohomología de álgebras de Lie y las álgebras conformes.

Desde 1993, la FaMAF ocupa un edificio en la Ciudad Universitaria inaugurado el 9 de marzo que alberga a docentes de Matemática, Física y

Computación. Este edificio fue ampliado en tres etapas: 1994, 2001 y 2010. En la planta baja se encuentra una muy buena biblioteca-hemeroteca de Matemática que incluye actualmente más de 10.000 libros, 80 títulos de revistas especializadas completas hasta 2005 y más de 120 suscripciones electrónicas anuales. La misma ha recibido aportes de la Universidad, de la Biblioteca Electrónica de la SECYT, del CONICET a través del CIEM y de los subsidios de los investigadores.

A partir de 1985 fue muy beneficiosa la relación de la FaMAF con el International Center for Theoretical Physics (ICTP) en Trieste. Varios matemáticos de nuestra Facultad fueron miembros asociados (Associate Members) y lo visitaron regularmente durante más de veinte años, participando en sus excelentes talleres y conferencias. Esto permitió crear contactos con matemáticos del mundo en desarrollo e incrementar la comunicación con científicos de Europa y Estados Unidos.

La labor de investigación fue apuntalada desde los inicios por la visita de investigadores del exterior que ayudaron a fortalecer los grupos de trabajo y ampliar las líneas de investigación. En su mayoría realizaron trabajos en colaboración o supervisaron estadas posdoctorales de matemáticos cordobeses.

Con el tiempo, la actividad científica se fue diversificando, lo que originó la formación de nuevos grupos. A los originales se agregaron los de Semántica Algebraica y Teoría de Números. La producción de trabajos de investigación en revistas internacionales reconocidas mantuvo un sólido crecimiento en cantidad y calidad. Es de destacar que en 1994, regresa a Córdoba el Dr. Aroldo Kaplan, distinguido especialista en geometría compleja quien realiza colaboraciones con varios colegas de la Facultad. En la actualidad las áreas de especialización en matemática cubren la mayor parte de las áreas de las veinte sesiones de los Congresos Internacionales de Matemática, a saber: Lógica, Álgebra, Teoría de Números, Geometría Diferencial, Geometría Compleja, Topología, Grupos y Álgebras de Lie, Análisis Armónico, Ecuaciones

Diferenciales Parciales, Estadística, Análisis Numérico, Optimización, Matemática Aplicada, Educación Matemática.

La docencia ha sido desde sus orígenes una preocupación central de los investigadores de la FaMAF. La planta docente de la Sección Matemática cuenta con más de setenta profesores con Dedicación Exclusiva (incluyendo CONICET), entre Titulares, Asociados y Adjuntos y más de cuarenta Auxiliares de la Docencia. Se dictan los cursos de Matemática para los estudiantes de las Licenciaturas en Matemática, Astronomía, Física y Computación y del Profesorado en Matemática y Postítulo en Matemática.

En 1992 se creó la Licenciatura en Computación, la que ha contribuido significativamente a aumentar el caudal de estudiantes y al desarrollo de los Laboratorios de Computación. Además, durante años la Facultad ha dictado cursos regulares de Matemática a las Facultades de Ciencias Químicas y Ciencias Económicas.

En 1990 se crearon las carreras de Profesorado en Matemática y en Física para atender la formación de profesores para la escuela media. El Profesorado en Matemática es una carrera de grado de cuatro años de duración, con una fuerte formación disciplinar complementada con sólidos trayectos de formación pedagógica y profesional. Esta carrera está respaldada en la Sección Matemática y en el Grupo de Investigación en Educación Matemática de la Facultad. En 1998 se creó en la Universidad Nacional de Córdoba un Magíster en Estadística Aplicada, cuya organización estuvo a cargo de las Facultades de Ciencias Económicas, Ciencias Agropecuarias y la FaMAF. El dictado de los cursos está a cargo de docentes de las tres Facultades y de otros docentes contratados. Se realizan también asesoramientos estadísticos, principalmente en el área de biología y medicina. Asimismo, existe una Maestría en Análisis y Procesamiento de Imágenes, organizada por la FaMAF y la FCEFyN.

Como parte del esfuerzo para el desarrollo de la matemática en la UNC, se ha puesto énfasis desde los comienzos en realizar actividades de coopera-

ción e integración con otros centros del país y de la región. Por ejemplo, desde 1970 y hasta 1994 se organizó bianualmente el Seminario Nacional de Matemática (11 veces), que contaron con amplia participación de matemáticos y estudiantes de otros centros de nuestro país y de la región. Con posterioridad, hubo bianualmente congresos por área, en diversas áreas: Álgebra (elENA), Geometría y Grupos de Lie (EGeo), Análisis, Matemática Aplicada y Estadística.

En colaboración con el International Center of Theoretical Physics (Trieste) se han organizado importantes congresos internacionales (Geometry and Lie Group Representations, 1989, 1995 y 2007) y en Hopf Algebras and Quantum Groups (2008) organizados parcial o íntegramente por matemáticos de la FaMAF, los cuales dieron lugar a publicaciones en las series Progress in Mathematics (Birkhäuser) y en Contemporary Mathematics (American Math. Society). Para su realización se contó con el apoyo financiero de instituciones nacionales, tales como CONICET, CONICOR y MINCyT, e internacionales como el ICTP (Trieste), la Fundación Antorchas, la National Science Foundation (U.S.A), ECOS (CNRS, Francia) y CIMPA (Francia).

En particular, en el congreso de 1989 se distinguieron con el título de Dr Honoris Causa de la UNC a varios destacados matemáticos: B. Kostant (MIT), G. Lusztig (MIT), W. Schmid (U. Harvard), D. Vogan (MIT), N. Wallach (U. Rutgers) y J. A. Wolf (U. C. Berkeley). Asimismo, mencionamos especialmente la Escuela Latinoamericana de Matemática (Análisis y Matemática Aplicada) realizada en Tanti, sierras de Córdoba, en 1991, con más de 200 participantes y con la presencia de importantes matemáticos de todo el mundo (L. Caffarelli (U. Minnesota), C Kenig (Chicago), F. Ricci (Instituto Politécnico de Turín) entre otros). Cabe también mencionar el Coloquio Latinoamericano de Álgebra que tuvo lugar en La Falda entre el 6 y el 10 de agosto de 2001. Participó en este encuentro una constelación de estrellas mundiales: B. Kostant (MIT), N. Wallach (UCSD), C. De Con-

cini (Roma), V. Kac (MIT), P. Etingof (MIT), F. Grunewald (Düsseldorf), E. B. Vinberg (Moscow State University), M. Kontsevich (U.C. Berkeley), J. A. Wolf (U. C. Berkeley), entre otros.

Finalmente, cabe destacar varios importantes congresos internacionales realizados más recientemente: Escuela Latinoamericana de Matemática (Mayo 2011, Esc. CIMPA), Encuentro Latinoamericano de Geometría Algebraica (ELGA, agosto de 2011), Galois representations and automorphic forms (agosto de 2011), Congreso Latinoamericano de Matemáticos (agosto de 2012) organizado por UMALCA en Córdoba, en simultáneo con la reunión anual de la Unión Matemática Argentina.

La FaMAF también estuvo presente en la fundación de UMALCA, la Unión Matemática de Latinoamérica y el Caribe, fundada en 1995 en Rio de Janeiro. Su creación fue resuelta el 27 de julio de 1995 en reunión en el Instituto de Matemática Pura e Aplicada (IMPA) con la participación de los Presidentes de las Sociedades Nacionales de Argentina (J. A. Tirao), Brasil, Chile, Colombia, Cuba, México, Uruguay y Venezuela y un representante de la de Perú.

El número total de doctores en matemática egresados de la Universidad Nacional de Córdoba desde 1985 asciende a 135. Para contextualizar su significado consignamos: En el mundo se gradúan por año aproximadamente 4.000 doctores. Si consideramos el promedio de graduados por año y por población de los países y normalizamos este índice a uno, entonces el número que le corresponde a Estados Unidos es 12,5 (unos 2.000 doctores), Inglaterra 10 (400), Israel 10 (40) y Argentina 0,8 (20).

El número de investigadores del CIEM ha llegado en la actualidad a setenta y el número de becarios a cuarenta y uno, mientras que en el 2012 eran más de sesenta, provenientes de otros centros del país (Tucumán, Tandil, Salta, La Plata, Bahía Blanca, San Luis, Santiago del Estero, Santa Fe, Rosario y Corrientes) y del exterior (Bolivia, Brasil y Colombia).

Creemos poder decir con satisfacción que en Córdoba hay un centro de Matemática de excelente nivel, reconocido como uno de los mejores de Latinoamérica.

Varios de sus investigadores han obtenido premios nacionales e internacionales. Por ejemplo, el premio Ramanujan de la International Mathematical Union en 2007 fue otorgado al Dr. Jorge Lauret y el Premio UMALCA 2020 a Ivan Angiono. La mayoría de los doctores formados en Córdoba han dirigido tesis doctorales y algunos son científicos destacados, en particular mencionamos a Nicolás Andruskiewitsch, Carlos Olmos y Jorge Lauret. Otra señal positiva es que en este momento hay 8 matemáticos de Córdoba que son miembros de la Academia Nacional de Ciencias (ANC), cronológicamente: Juan A. Tirao, Cristián U. Sánchez, Roberto J. Miatello, Jorge A. Vargas, Isabel G. Dotti, Aroldo Kaplan, Nicolás Andruskiewitsch, Carlos E. Olmos y Jorge Lauret.

En base a los conceptos y datos vertidos anteriormente, podemos afirmar que es mucho lo avanzado en el desarrollo matemático en los sesenta y siete años transcurridos desde la creación del IMAF. En nuestra opinión esto ha sido posible por el compromiso y determinación de los iniciadores, por el trabajo en equipo del grupo todo y por el esfuerzo de la primera generación de doctores que estupendamente apuntaló la labor inicial. Puede apreciarse que existe un sólido centro matemático, un grupo de investigadores reconocidos, un número creciente de estudiantes de doctorado del

país y del exterior y una conexión con los grupos matemáticos del país, de la región y de importantes centros del hemisferio norte.

Ahora bien, tomando el estado actual como punto de partida, el gran desafío sería encaminarse hacia más ambiciosos horizontes, buscando alcanzar un nivel similar al de los centros que lideran la investigación matemática a nivel mundial. Como pasos convenientes para seguir progresando, creemos que sería valioso realizar incorporaciones en áreas aun vacantes o poco representadas, tales como geometría algebraica, ecuaciones en derivadas parciales, ciertos temas de matemática aplicada y teoría de números.

Sería muy útil aumentar el flujo de investigadores visitantes y la participación en congresos internacionales para seguir siendo fuertes en las áreas más activas en Córdoba. En cuanto a la docencia, sería saludable ver en años futuros una mayor presencia de matemáticos de FaMAF en otras carreras para incrementar la conexión con otras facultades, contribuyendo al mejoramiento de la enseñanza de la Matemática y a la ocupación de los matemáticos que egresen de FaMAF.

Finalmente, con respecto a la vinculación con el medio, sería deseable que se impulse un sano contacto con empresas del medio para que crezca el número de nuestros egresados que hagan aportes científicos en el ámbito tecnológico.

Finalmente les digo que espero reencontrarlos en diez años más sabios y buenos. Muchas gracias.

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WHAT IS A QUANTUM GROUP?

NICOLÁS ANDRUSKIEWITSCH

ABSTRACT. Formal deformations (or quantizations) of Poisson manifolds and Hopf algebras converge in the notion, due to Drinfeld, of quantum group, which is introduced after a brief overview of these topics. In short, this is an invitation to [10]. Some further developments and applications are discussed.

This is the revised text of the conference given on the occasion of the 40th anniversary of CIEM. Due to space restrictions, many statements are imprecise and the bibliography is incomplete.

1. HOPF ALGEBRAS

In 1881, the Norwegian mathematician Sophus Lie made a breakthrough by founding the (now called) Lie theory. In today's parlance, the basic points are:

- A *Lie group* is a group G that is also a differential variety such that the maps $m : G \times G \rightarrow G$ (multiplication) and $\iota : G \rightarrow G$ (inversion) are morphisms of differential varieties, i.e., smooth.
- A *Lie algebra* is a pair $(\mathfrak{g}, [\cdot, \cdot])$, where $\mathfrak{g}$ is a vector space and $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ is a bilinear antisymmetric map (the *Lie bracket*) that satisfies the Jacobi identity.
- *The Dictionary:* If G is a Lie group, then $\mathfrak{g} := \mathcal{T}_e G$ (the tangent space at the identity) admits a structure of Lie algebra. Many important properties of G can be expressed in terms of $\mathfrak{g}$ (which means that we can go back and forth between Algebra and Geometry).

These definitions allow for many variations: real Lie groups, complex Lie groups, algebraic groups, etc.¹ For the algebraic groups, the Dictionary still

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¹Lie himself dealt with complex varieties. See [13, 6] for historical accounts.

²A parallel development was started by Armand Borel in order to extend fundamental results of H. Hopf on the cohomology of Lie groups; see the discussion in [1]. The natural description of commutative Hopf algebras as algebras of functions on proalgebraic groups appeared later.

holds in characteristic 0 but it fails in positive characteristic as observed by Chevalley. To remedy this, Dieudonné proposed in 1954 to consider, instead of the Lie algebra $\mathfrak{g}$ of an algebraic group G , its hyperalgebra $\mathcal{U}(G)$ (that can be thought of as the algebra of distributions with support in the identity). In characteristic 0, $\mathcal{U}(G) \simeq U(\mathfrak{g})$ but in positive characteristic this is no longer true; the key new feature are the divided powers. In 1956, a young Pierre Cartier presented a formalism for the hyperalgebras of Dieudonné, this is essentially the birth of the theory of Hopf algebras².

2. QUANTIZATION.

A Poisson algebra is a commutative algebra R which is also a Lie algebra with Lie bracket $\{ \cdot, \cdot \} : R \otimes R \rightarrow R$ that satisfies the Leibniz identity

$$(2.1) \quad \{f, gh\} = \{f, g\}h + g\{f, h\}, \quad \forall f, g, h \in R.$$

We also say that $\{ \cdot, \cdot \}$ is a Poisson bracket. A (real or complex or ...) manifold M is *Poisson* if $C^\infty(M)$ (or the corresponding algebra of functions) is Poisson.

Example 2.1. The polynomial ring $R_{2n} := \mathbb{C}[p_1, \dots, p_n, q_1, \dots, q_n]$ is a Poisson algebra with the bracket defined on the variables by

$$(2.2) \quad \{p_i, p_j\} = 0, \quad \{q_i, q_j\} = 0, \quad \{p_i, q_j\} = \delta_{ij}, \\ i, j \in \mathbb{I}_n := \{1, 2, \dots, n\},$$

and extended by the Leibniz rule (2.1).

Example 2.2. A symplectic manifold is Poisson; for instance the cotangent bundle of any manifold is a symplectic manifold, hence it is Poisson. A concrete example is $\mathbb{C}^{2n} \simeq \mathcal{T}^*(\mathbb{C}^n)$; the Lie bracket turns out to be determined by (2.2).

Example 2.3. The symmetric algebra $S(\mathfrak{g}) \simeq \mathcal{O}(\mathfrak{g}^*)$ of a Lie algebra $\mathfrak{g}$ is a Poisson algebra with the bracket $\{, \}$ induced by that of $\mathfrak{g}$. Also, when $\mathfrak{g}$ is real, $C^\infty(\mathfrak{g}^*)$ is a Poisson algebra. In other words $\mathfrak{g}^*$ is a Poisson manifold, but it is not symplectic.

Theorem 2.4. [15, 24] *If M is a Poisson manifold and $p \in M$, then there exists a maximal symplectic submanifold $L \ni p$ (generated by the trajectories of the Hamiltonian fields $H_f = \{f, \}$ through p). Thus M is foliated by symplectic leaves.*

Example 2.5. The symplectic leaves of $\mathfrak{g}^*$ are the coadjoint orbits (of the representation of the Lie group $\text{Ad } \mathfrak{g}$ on $\mathfrak{g}^*$).

Intuitively, the mathematical notion of quantization consists, first, in replacing a Poisson algebra by a non-commutative one, and second, in studying the (unitary) representations of the latter. One of the possible formulations of the first stage is the so-called *deformation quantization*, see [4] for details and historical references.

Definition 2.6. A *formal deformation* of a (complex) Poisson algebra R is an (associative) $\mathbb{C}[[h]]$ -algebra A with multiplication $\star$ that satisfies

$$(2.3) \quad A/hA \simeq R, \quad \text{as } \mathbb{C}\text{-algebra,}$$

$$(2.4) \quad A \simeq R[[h]], \quad \text{as } \mathbb{C}[[h]]\text{-module,}$$

$$(2.5) \quad \{f, g\} = \frac{[f, g]}{h} = \frac{f \star g - g \star f}{h} \pmod{h},$$

for all $f, g \in R$.

The formal deformation A is also called the *quantization* of R whereas R is the *classical limit* of A . Notice that the bracket $\{, \}$ defined by (2.5) is Poisson.

There are variations of this definition, e.g. polynomial deformations in which the algebra $\mathbb{C}[h]$ is taken instead of $\mathbb{C}[[h]]$.

Example 2.7. A quantization of $R_{2n} = \mathbb{C}[p_i, q_j : i, j \in \mathbb{I}_n]$ with the Poisson bracket (2.2) is the Weyl algebra $A_{2n} := \mathbb{C}[h]\langle x_1, \dots, x_n, y_1, \dots, y_n \rangle$ with defining relations

$$x_i x_j = x_j x_i, \quad y_i y_j = y_j y_i, \quad x_i y_j - y_j x_i = h \delta_{ij},$$

$i, j \in \mathbb{I}_n.$

Example 2.8. A quantization of the symmetric algebra $S(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is the enveloping algebra $U(\mathfrak{g}_h)$, where $\mathfrak{g}_h = \mathfrak{g}[[h]]$ with bracket $h[\cdot, \cdot]$.

Problem. If M is a Poisson manifold, does $C^\infty(M)$ admit a formal deformation?

A first substantial answer is the following:

Theorem 2.9. [7] *If M is symplectic, then $C^\infty(M)$ admits a formal deformation.*

The complete answer is a famous result of Kontsevich.

Theorem 2.10. [17] *If M is Poisson, then $C^\infty(M)$ admits a formal deformation.*

As for the second stage of quantization, there is an heuristic principle:

If A is a quantization of $C^\infty(M)$, then the irreducible representations of A are parametrized (at least governed) by the symplectic leaves of M .

This principle extends another called the *orbit method*, see [16]:

Given a finite-dimensional Lie algebra $\mathfrak{g}$, the irreducible representations of the Lie group $G = \text{Ad } \mathfrak{g}$, or of $U(\mathfrak{g})$, are parametrized by the coadjoint orbits of $\mathfrak{g}$.

Strictly speaking both principles fail; however, they provide efficient roadmaps.

3. POISSON-LIE GROUPS

The next Definitions and Theorem are due to Drinfeld; see e.g. [10].

Definition 3.1. A *Poisson-Lie group* is a Lie group G that is also a Poisson manifold such that the multiplication $m : G \times G \rightarrow G$ and the inversion $\iota : G \rightarrow G$ are morphisms of Poisson manifolds.

Definition 3.2. A finite-dimensional *Lie bialgebra* is a pair $(\mathfrak{g}, \delta)$, where $\mathfrak{g}$ is a Lie algebra, $\dim \mathfrak{g} < \infty$, and $\delta : \mathfrak{g} \rightarrow \Lambda^2(\mathfrak{g})$ is a 2-cocycle³ such that $(\mathfrak{g}^*, {}^t\delta)$ is a Lie algebra (then $\mathfrak{g}^*$ becomes a Lie bialgebra).

Here is a ‘Poisson’ version of the Dictionary:

Theorem 3.3. • If G is a Poisson-Lie group, then $\mathfrak{g} = \text{Lie } G$ admits a Lie bialgebra structure.

- If $(\mathfrak{g}, \delta)$ is a Lie bialgebra, then the connected simply connected Lie group G with $\mathfrak{g} \simeq \text{Lie } G$ admits a Poisson-Lie group structure.

Analogously one may consider a Poisson (affine) algebraic group G ; in this case the algebra of functions $\mathcal{O}(G)$ is a Hopf-Poisson algebra.

We are led to study Lie bialgebras. Among them, the quasitriangular ones are important: a Lie bialgebra $(\mathfrak{g}, \delta)$ is quasitriangular if there exists $r = \sum_i r_i \otimes r^i \in \mathfrak{g} \otimes \mathfrak{g}$, such that $\delta = \partial r$ and r satisfies the classical Yang-Baxter equation:

$$[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0,$$

where $r^{13} = \sum_i r_i \otimes 1 \otimes r^i$, etc. Also, if $r + r^{21} \in S^2 \mathfrak{g}$ defines a nondegenerate inner product on $\mathfrak{g}^*$, then $(\mathfrak{g}, \delta)$ is *factorizable*. Here are fundamental examples. We fix

- a finite-dimensional complex simple Lie algebra $\mathfrak{g}$,
- a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and
- a choice of a set of simple roots $\Pi \subset \mathfrak{h}^*$.

Theorem 3.4. [5] *The factorizable Lie bialgebras $(\mathfrak{g}, \delta)$ are classified by*

- a discrete parameter (I_1, I_2, ν) where $I_1, I_2 \subset \Pi$, $\nu : I_1 \rightarrow I_2$ is a bijective morphism of diagrams and for all $x \in I_1$, $\nu^n(x)$ is not defined for some n ;
- a continuous parameter $\lambda \in \Lambda^2(\mathfrak{h})$ that satisfies certain linear conditions depending on the discrete parameter.

The bracket corresponding to $I_1 = I_2 = \emptyset$, $\lambda = 0$ is called the Sklyanin bracket.

A classical result says that semisimple Lie algebras are rigid, i.e. non deformable; but this not the case at the level of enveloping algebras.

Definition 3.5. [10] A quantization of a Lie bialgebra $(\mathfrak{g}, \delta)$ is a $\mathbb{C}[[h]]$ -Hopf algebra $\mathcal{U}$ that satisfies

$$\mathcal{U}/h\mathcal{U} \simeq U(\mathfrak{g}), \quad \text{as } \mathbb{C}\text{-Hopf algebra,}$$

$$\mathcal{U} \simeq U(\mathfrak{g})[[h]], \quad \text{as } \mathbb{C}[[h]]\text{-module,}$$

$$\delta(f) = \frac{\Delta(f) - \Delta^{\text{op}}(f)}{h} \mod h, \quad \text{for all } f \in \mathfrak{g}.$$

Remark 3.6. The finite dual of $\mathcal{U}$ is a Hopf algebra that is, roughly, a quantization of the Poisson-Lie group associated a $(\mathfrak{g}, \delta)$.

Theorem 3.7. [11] *Every Lie bialgebra is quantizable, i.e. admits a quantization.*

Brief historical note. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra.

- In 1981, Kulish and Reshetikin defined a $\mathbb{C}[[h]]$ -algebra $U_h(\mathfrak{sl}_2(\mathbb{C}))$ by generators and relations; the Hopf algebra structure was described a little later by Sklyanin.
- In 1983, Drinfeld defined by generators and relations a $\mathbb{C}[[h]]$ -Hopf algebra $U_h(\mathfrak{g})$ that quantizes the *Sklyanin bracket*.
- Independently, Jimbo defined by generators and relations a $\mathbb{C}$ -Hopf algebra $U_q(\mathfrak{g})$ where $q \in \mathbb{C}^\times \setminus \mathbb{G}_\infty$; it is related to $U_h(\mathfrak{g})$ by the (formal) convention $q = \exp h$. Here and below $\mathbb{G}_\infty$ denotes the group of roots of 1.

³In general δ should satisfy the co-Jacobi identity, which makes sense also for $\dim \mathfrak{g} = \infty$.

- Later Woronowicz defined a C^* -algebra $\mathcal{O}_q(SU(2))$ for $q \in \mathbb{R}, q \neq 0, 1, -1$. In a proper sense this is a *compact form* of the quantization of the Poisson-Lie group $SL_2(\mathbb{C})$ respect to the bracket of Sklyanin.
- The definition extends to a C^* -algebra $\mathcal{O}_q(U)$ for any almost simple compact Lie group U , $q \in \mathbb{R}, q \neq 0, 1, -1$.

Theorem 3.8. [23, 18] *The irreducible unitary representations of $\mathcal{O}_q(U)$ are in bijective correspondence with the symplectic leaves of the Poisson-Lie group U .*

The Poisson structure on U quantized by $\mathcal{O}_q(U)$ corresponds to the Sklyanin bracket; for other brackets from Theorem 3.4, the representation theory of the relevant compact quantum group is loosely related to the symplectic leaves [18].

4. WHAT IS A QUANTUM GROUP?

As we know, some classes of groups can be described in terms of Hopf algebras:

- An affine algebraic group is the spectrum of a finitely generated commutative Hopf algebra.⁴
- A compact group is the spectrum of a commutative unital C^* -Hopf algebra.

One may speculate that a quantum group should be (formally) the spectrum of a Hopf algebra quantization of a Poisson-Lie group. But Drinfeld goes further:

Definition 4.1. [10] The category of quantum groups is the opposite to the category of Hopf algebras with bijective antipode.

Zoo of quantum groups. Here is a list of important examples of quantum groups that have been studied intensively. Let $\mathfrak{g}$ be a finite-dimensional simple complex Lie algebra and let G be a connected Lie group with $\mathfrak{g} = \text{Lie } G$.

- [10, 14] The quantized enveloping algebra $U_q(\mathfrak{g})$, $q \in \mathbb{C}^\times \setminus \mathbb{G}_\infty$.

⁴This is in characteristic 0; otherwise this is an affine group scheme.

- [9] The version of $U_q(\mathfrak{g})$ where $q \in \mathbb{G}_\infty$ (with restrictions on the order of q).
- [20] The quantum divided powers algebra $\mathcal{U}_q(\mathfrak{g})$, $q \in \mathbb{G}_\infty$ (restrictions on $\text{ord } q$).
- [20] The (finite-dimensional) small quantum group $\mathfrak{u}_q(\mathfrak{g})$, $q \in \mathbb{G}_\infty$ (with restrictions on $\text{ord } q$).
- [12, 20, 22] The quantized algebra of functions $\mathcal{O}_q(G)$, $q \in \mathbb{C}^\times$ (with restrictions on $\text{ord } q$). For q real, the compact forms $\mathcal{O}_q(U)$ [25].

There are versions of $U_q(\mathfrak{g})$, $\mathcal{U}_q(\mathfrak{g})$, $\mathfrak{u}_q(\mathfrak{g})$ or $\mathcal{O}_q(G)$ for $\mathfrak{g}$ a Kac–Moody algebra, or a (contragredient) Lie superalgebra, or a modular Lie (super)algebra, etc.

5. CLASSIFICATIONS

Drinfeld wrote in [10]: *I believe that most of the examples of noncommutative noncocommutative Hopf algebras invented independently of integrable quantum system theory are counterexamples rather than ‘natural’ examples.* In this regard, we mention:

Theorem 5.1. [2] *Let H be a pointed Hopf algebra with abelian group, reductive, with finite Gelfand-Kirillov dimension, satisfying some extra conditions. Then H is a variation of $U_q(\mathfrak{g})$, $q \in \mathbb{C}^\times \setminus \mathbb{G}_\infty$.*

Theorem 5.2. [3] *Let H be a pointed Hopf algebra with abelian group, $\dim H < \infty$, satisfying some extra conditions. Then H is a variation of $\mathfrak{u}_q(\mathfrak{g})$, $q \in \mathbb{G}_\infty$.*

6. APPLICATIONS AND REPRESENTATIONS

Hopf algebras (quantum groups) have applications to knot invariants, subfactors (von Neumann algebras), conformal field theory... that go through their representations. We summarize the main features. Let H be a Hopf algebra.

- The category $\text{rep } H$ of finite-dimensional representations is a tensor one: if $M, N \in \text{rep } H$, then

$M \otimes N \in \text{rep } H$ with action

$$h \cdot (m \otimes n) := \Delta(h) \cdot (m \otimes n), \\ h \in H, m \in M, n \in N.$$

- [10] One says that H is *quasitriangular* if there is a suitable natural isomorphism

$$c_{M,N} : M \otimes N \xrightarrow{\sim} N \otimes M, \quad M, N \in \text{rep } H.$$

The natural transformation c (that generalizes the usual flip) is called the *braiding* since it gives rise to representations of the braid groups.

- [10] When $\dim H < \infty$, H is embedded in a quasitriangular Hopf algebra $D(H)$ ($\simeq H \otimes H^*$ as vector spaces) called the Drinfeld double of H ⁵
- The quantum groups $U_q(\mathfrak{g})$, $\mathcal{U}_q(\mathfrak{g})$, $\mathfrak{u}_q(\mathfrak{g})$ are quasitriangular.

These considerations led to the study of the representation theories of the various quantum groups listed above. This was undertaken by many authors; we mention:

- [19, 21] $\text{rep } U_q(\mathfrak{g})$, $q \in \mathbb{C}^\times \setminus \mathbb{G}_\infty$, is closely related to $\text{rep } U(\mathfrak{g})$ (but it has a very different braiding).
- [9, 8, 18] $\text{rep } U_q(\mathfrak{g})$, $\text{rep } \mathcal{O}_q(G)$, $q \in \mathbb{G}_\infty$, are governed by the symplectic leaves of the Poisson-Lie groups G^d and G . Here G^d is connected and $\text{Lie } G^d \simeq \mathfrak{g}^*$.
- [20] $\text{rep } \mathfrak{u}_q(\mathfrak{g})$, $\text{ord } q = p$ prime is analogous to $\text{rep } \mathfrak{u}(\mathfrak{g})$, in char. $p > 0$.

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⁵When $\dim H = \infty$, the Yetter-Drinfeld modules are considered instead of the Drinfeld double.

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ROOT SYSTEMS AND HOPF ALGEBRAS

IVÁN ANGIOÑO

ABSTRACT. We survey the notions of Nichols algebras of diagonal type, the associated root systems, the motivation from Lie Theory and the connection of this combinatorial object with the Gelfand-Kirillov dimension of Nichols algebras of diagonal type.

1. INTRODUCTION

The purpose of this expository article is to recall the last advances towards the relation between Hopf algebras of finite growth and root systems. We start from basic definitions, recalling classical objects as semisimple Lie algebras, their associated root systems, the axiomatization of this notion and the further generalizations for other objects in Lie Theory. Let $\mathbb{k}$ be an algebraically closed field of characteristic zero. All vector spaces, tensor products, etc., considered here are over $\mathbb{k}$.

2. ROOT SYSTEMS IN (CLASSICAL) LIE THEORY

We start by recalling from [8] the classical notions of semisimple Lie algebras and root systems, together with crucial results on these topics. Later on, we recall from [11] the analogous objects for Lie superalgebras and finish with the PBW Theorem for the associated universal enveloping algebras.

2.1. Semisimple Lie algebras. A *Lie algebra* is a vector space $\mathfrak{g}$ together with a linear map $[\cdot, \cdot] : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ (the *bracket*) which is antisymmetric and satisfies the Jacobi identity: that is,

$$[x, y] = -[y, x], \quad [x, [y, z]] = [[x, y], z] + [y, [x, z]],$$

for all $x, y, z \in \mathfrak{g}$. A Lie subalgebra of $\mathfrak{g}$ is a subspace $\mathfrak{h}$ such that $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$, so $\mathfrak{h}$ itself is a Lie algebra.

A way to produce examples (and to move from the associative world to the one of Lie algebras) is to take an associative algebra A and define the bracket $[\cdot, \cdot] : A \otimes A \rightarrow A$ as $[x, y] = xy - yx$ for all $x, y \in A$. In particular:

- For $A = \mathbb{k}^{n \times n}$, the associated Lie algebra is denoted by $\mathfrak{gl}(n)$.
- The subspace $\mathfrak{sl}(n)$ of matrices of trace 0 is a Lie subalgebra of $\mathfrak{gl}(n)$.

An ideal of $\mathfrak{g}$ is a subspace $\mathfrak{I} \subseteq \mathfrak{g}$ such that $[\mathfrak{I}, \mathfrak{g}] \subseteq \mathfrak{I}$. A Lie algebra $\mathfrak{g}$ is *simple* if $\dim \mathfrak{g} > 1$ and the unique ideals are the trivial ones, 0 and $\mathfrak{g}$. A Lie algebra is *semisimple* if it is the direct sum of simple Lie algebras.

The classification of all finite dimensional complex Lie algebras is well known: it involves some subalgebras of $\mathfrak{gl}(n)$ together with a few exceptions. More precisely:

Theorem 2.1 (Killing, Cartan). *Let $\mathbb{k} = \mathbb{C}$. A finite dimensional Lie algebra $\mathfrak{g}$ is simple if and only if $\mathfrak{g}$ is (isomorphic to) $\mathfrak{sl}(n)$, $\mathfrak{so}(n)$, $\mathfrak{sp}(n)$, or the exceptional Lie algebras E_k , $k = 6, 7, 8$, F_4 , G_2 of dimensions 72, 133, 248, 52 and 14.*

A way to obtain and easily describe this classification is by means of the root system (and the associated Cartan matrix). Let us motivate it from an example.

Example 2.2. The Lie algebra $\mathfrak{g} = \mathfrak{sl}(n+1)$ has a *triangular* decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ as a vector space, where $\mathfrak{n}_-$, respectively $\mathfrak{h}$, $\mathfrak{n}_+$ is the subalgebra of lower triangular, respectively diagonal, upper triangular, matrices:

- (1) The set $\{h_i = E_{ii} - E_{i+1, i+1} | 1 \leq i \leq n\}$ is a basis of $\mathfrak{h}$.
- (2) The set $\{E_{i, j+1} | 1 \leq i \leq j \leq n\}$ is a basis of $\mathfrak{n}_+$. As a Lie algebra, $\mathfrak{n}_+$ is generated by

- $\{e_i := E_{i,i+1} | 1 \leq i \leq n\}$ (i.e. any element of $\mathfrak{n}_+$ is a linear combination of brackets of e_i .)
- (3) Analogously, the set $\{E_{j+1,i} | 1 \leq i \leq j \leq n\}$ is a basis of $\mathfrak{n}_-$, and $\mathfrak{n}_-$ is generated by $\{f_i := E_{i+1,i} | 1 \leq i \leq n\}$
- (4) The action of $\mathfrak{h}$ is diagonalizable, that is $\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$, where for each $\alpha \in \mathfrak{h}^*$ we set

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : [h, x] = \alpha(h)x, \forall h \in \mathfrak{h}\}.$$

The *root system* is the subset Δ of all $\alpha \in \mathfrak{h}^* - \{0\}$ such that $\mathfrak{g}_\alpha \neq 0$. Hence, $\mathfrak{g}_0 = \mathfrak{h}$, and $\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha)$.

- (5) Let $A = (a_{ij}) \in \mathbb{Z}^{n \times n}$ be the matrix such that

$$a_{ij} = \begin{cases} 2, & i = j, \\ -1, & |i - j| = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\alpha_i \in \mathfrak{h}^*$ be such that $\alpha_j(h_i) = a_{ij}$, and $\alpha_{ij} := \sum_{i \leq k \leq j} \alpha_k$. Then

$$\mathfrak{g}_{\alpha_{ij}} = \mathbb{C}E_{ij+1}, \quad \mathfrak{g}_{-\alpha_{ij}} = \mathbb{C}E_{j+1i}, \quad 1 \leq i, j \leq n.$$

Thus $\Delta = \Delta_+ \cup \Delta_-$, where

$$\Delta_+ = \{\alpha_{ij} | 1 \leq i, j \leq n\}, \quad \Delta_- = -\Delta_+.$$

We have a similar picture in general: Every semisimple Lie algebra has a *triangular* decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ as a vector space, where

- (i) $\mathfrak{h}$ is a maximal abelian subalgebra called the Cartan subalgebra.
- (ii) $\mathfrak{n}_+$ is generated by a set $\{e_i | 1 \leq i \leq n\}$ and $\mathfrak{n}_-$ is generated by a set $\{f_i | 1 \leq i \leq n\}$, in such a way that $\{h_i := [e_i, f_i] | 1 \leq i \leq n\}$ is a basis of $\mathfrak{h}$ and $\{e_i, h_i, f_i\}$ is isomorphic to $\mathfrak{sl}(2)$ for all $1 \leq i \leq n$.
- (iii) The action of $\mathfrak{h}$ is diagonalizable, that is $\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$.
- (iv) The *root system* is the subset Δ of all $\alpha \in \mathfrak{h}^* - \{0\}$ such that $\mathfrak{g}_\alpha \neq 0$. Again, $\mathfrak{g}_0 = \mathfrak{h}$, and $\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha)$, $\dim \mathfrak{g}_\alpha = 1$ for all $\alpha \in \Delta$.

Remark 2.3. Let $A = (a_{ij})$ be the matrix such that $a_{ij} = \alpha_j(h_i)$, $1 \leq i, j \leq n$, where $\alpha_j \in \mathfrak{h}^*$ is such that $e_j \in \mathfrak{g}_{\alpha_j}$. A is called the *Cartan matrix* of $\mathfrak{g}$. Then:

- $a_{ij} \in \mathbb{Z}$, $a_{ii} = 2$, $a_{ij} \leq 0$ for all $i \neq j$.

- Let $\Delta_\pm = \Delta \cap (\bigoplus_{i=1}^n \mathbb{N}_0 \alpha_i)$. Then $\Delta = \Delta_+ \cup \Delta_-$, $\Delta_- = -\Delta_+$ and $\mathfrak{n}_\pm = \bigoplus_{\alpha \in \Delta_\pm} \mathfrak{g}_\alpha$.
- Let $\sigma_i : \mathfrak{h}^* \rightarrow \mathfrak{h}^*$ be the linear transformations $\sigma_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$. Then

$$\sigma_i^2 = \text{id}, \quad \sigma_i(\Delta_+ - \{\alpha_i\}) = \Delta_+ - \{\alpha_i\}.$$

That is, the σ_i 's are reflections fixing Δ .

2.2. Lie superalgebras. Analogously, we can define a notion of Lie algebra but in super vector spaces. More concretely, a Lie superalgebra is a $\mathbb{Z}_2$ -graded vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ together with a $\mathbb{Z}_2$ -graded linear map $[\cdot, \cdot] : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ such that

$$[x, y] = -(-1)^{|x||y|}[y, x],$$

$$[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]],$$

for all $x, y, z \in \mathfrak{g}$ homogeneous. Here, $|x| \in \mathbb{Z}_2$ is the $\mathbb{Z}_2$ -degree of the homogeneous element $x \in \mathfrak{g}$, $\mathfrak{g}_0$ is the even part and $\mathfrak{g}_1$ the odd part.

Subalgebras, ideals and simple Lie superalgebras are defined accordingly.

Example 2.4. As for Lie algebras, we can start from an associative $\mathbb{Z}_2$ -graded algebra $\mathfrak{g} = A_0 \oplus A_1$ and define the bracket $[x, y] = xy - (-1)^{|x||y|}yx$. Using this construction we have analogous examples of classical Lie algebras:

$$\mathfrak{gl}(m|n) = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbb{C}^{(m+n) \times (m+n)} : \right.$$

$$\left. A, D \text{ even}, B, C \text{ odd} \right\},$$

$$\mathfrak{sl}(m|n) = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathfrak{gl}(m|n) : \right.$$

$$\left. \text{tr } A - \text{tr } D = 0 \right\}.$$

Also, any Lie algebra $\mathfrak{g}$ is a Lie superalgebra with $\mathfrak{g} = \mathfrak{g}_0$ (i.e. $\mathfrak{g}_1 = 0$).

For most of the finite-dimensional simple Lie superalgebras over $\mathbb{C}$ we can find a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ as a vector space, with $\mathfrak{h}$ a maximal abelian subalgebra called the Cartan subalgebra. The action of $\mathfrak{h}$ is diagonalizable, that is $\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$. The *root system* is the subset Δ of all $\alpha \in \mathfrak{h}^* - \{0\}$ such that $\mathfrak{g}_\alpha \neq 0$. Again, $\mathfrak{g}_0 = \mathfrak{h}$, and $\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha)$, $\dim \mathfrak{g}_\alpha = 1$ for all $\alpha \in \Delta$.

The root system decomposes as $\Delta = \Delta_+ \cup \Delta_-$ such that $\Delta_- = -\Delta_+$ and $\mathfrak{n}_\pm = \bigoplus_{\alpha \in \Delta_\pm} \mathfrak{g}_\alpha$. But we have *even* roots when $\mathfrak{g}_\alpha \subset \mathfrak{g}_0$, and *odd* roots when $\mathfrak{g}_\alpha \subset \mathfrak{g}_1$.

2.3. Modules and enveloping algebras. As for associative algebras, we want to study modules for Lie algebras. A $\mathfrak{g}$ -module is a pair $(V, \cdot)$, where V is a $\mathbb{k}$ -vector super space and $\cdot : \mathfrak{g} \otimes V \rightarrow V$ is a linear map such that

$$[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v),$$

for all $x, y \in \mathfrak{g}, v \in V$.

Here $\cdot$ can be defined with a Lie algebra map $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ as usual:

$$\rho(x)(v) = x \cdot v, \quad \text{for all } x, y \in \mathfrak{g}, v \in V.$$

For example $V = \mathbb{k}^n$ is a $\mathfrak{gl}(n)$ -module in a natural way, and thus a $\mathfrak{sl}(n)$ -module as well. For $\mathbb{k} = \mathbb{C}$, a Lie algebra is semisimple if and only if every finite-dimensional module is completely reducible, i.e. every submodule admits a complementary submodule. But this fails for Lie superalgebras.

Anyway, the category of modules keeps fundamental information on the structure of the Lie (super) algebra. We may wonder if we can study an equivalent category but made of modules over an associative algebra. This is the idea behind the notion of the universal enveloping algebra.

Definition 2.5. Let $\mathfrak{g}$ be a Lie (super) algebra. A *universal enveloping algebra* is pair (U, ι) , where U is an associative (super) algebra and $\iota : \mathfrak{g} \rightarrow U$ is a Lie algebra map (where U is viewed as a Lie algebras as in Example 2.4), universal in the following sense: For any other pair (A, κ) as above, there exists a unique algebra map $\varphi : U \rightarrow A$ such that $\varphi \circ \iota = \kappa$.

Remark 2.6. As expected, the universal enveloping algebra does exist and is unique. It can be identified with the quotient

$$U(\mathfrak{g}) = T(\mathfrak{g}) / \langle x \otimes y - (-1)^{|x||y|} y \otimes x - [x, y], \\ x, y \in \mathfrak{g} \text{ homogeneous} \rangle$$

Due to the universal property, there exists a equivalence between the category of $\mathfrak{g}$ -modules and the category of $U(\mathfrak{g})$ -modules.

Let $\mathfrak{g}$ be a Lie algebra. Due to a result of Poincaré-Birkhoff-Witt we know a basis of $U(\mathfrak{g})$ starting from an ordered basis $(x_i)_{i \in I}$ of $\mathfrak{g}$, made by ordered products of the x_i 's with arbitrary powers. But for Lie superalgebras, if we take $x \in \mathfrak{g}_1$, then $[x, x] \in \mathfrak{g}_0$: moreover, in $U(\mathfrak{g})$ either $[x, x] = 0$ or $[x, x] = 2x^2$. Thus we do not need arbitrary powers of odd elements. Explicitly:

Theorem 2.7 (PBW Theorem). *Let $B = B_0 \cup B_1$ be a basis of $\mathfrak{g}$, where $B_i \subset \mathfrak{g}_i$ is a basis of $\mathfrak{g}_i$. Let $<$ be a total order on B . Then the set of monomials*

$$x_1^{a_1} \cdots x_n^{a_n}, \quad n \in \mathbb{N}_0, \quad x_1 < \cdots < x_n \in B, \\ a_i \in \mathbb{N} \text{ if } x_i \in B_0, \quad a_i = 1 \text{ if } x_i \in B_1,$$

is a basis of $U(\mathfrak{g})$. In particular, ι is injective.

3. NICHOLS ALGEBRAS AND ROOT SYSTEMS

Next we deal with Nichols algebras and their root systems. We recall the definition of this kind of algebras, with focus in the diagonal case, the associated root system and the relation between the finiteness of this set and the Gelfand-Kirillov dimension of the Nichols algebra.

3.1. Braided vector spaces. To define Nichols algebras, we need first to recall the definition of a braided vector space. This is a pair (V, c) , where V is a vector space and $c : V \otimes V \rightarrow V \otimes V$ is a linear automorphism satisfying the *braid equation*:

$$(c \otimes \text{id})(\text{id} \otimes c)(c \otimes \text{id}) = (\text{id} \otimes c)(c \otimes \text{id})(\text{id} \otimes c).$$

For example we have braided vector spaces of *diagonal type*: it means that there exists a matrix $\mathbf{q} = (q_{ij}) \in (\mathbb{k}^\times)^{\theta \times \theta}$ and a basis $\{x_i\}_{1 \leq i \leq \theta}$ of V such that

$$c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i, \quad \text{for all } 1 \leq i, j \leq \theta.$$

Braided vector spaces are related with *braid groups* $\mathbb{B}_n$ on n strands. Recall that $\mathbb{B}_n$ is the group presented by generators τ_i , $1 \leq i \leq n-1$,

and relations

$$\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}, \quad \tau_i \tau_j = \tau_j \tau_i, \quad |i - j| > 1.$$

Remark 3.1. Each braided vector space (V, c) induces a group homomorphism

$$\rho : \mathbb{B}_n \rightarrow \text{Aut}(V^{\otimes n}), \quad \tau_i \mapsto \text{id}_{V^{\otimes i-1}} \otimes c \otimes \text{id}_{V^{\otimes n-i-1}}.$$

In other words, $V^{\otimes n}$ is a $\mathbb{B}_n$ -module.

The presentation of $\mathbb{B}_n$ above resembles the one of the symmetric group $\mathbb{S}_n$, since it is generated by the reflections $\sigma_i = (i \ i+1)$, $1 \leq i \leq n-1$, which satisfy the relations above, but the σ_i 's also satisfy that $\sigma_i^2 = e$. Thus we have a group homomorphism $\pi : \mathbb{B}_n \twoheadrightarrow \mathbb{S}_n$ such that $\tau_i \mapsto \sigma_i$, $1 \leq i \leq n-1$.

There exists a set-theoretical section $s : \mathbb{S}_n \hookrightarrow \mathbb{B}_n$, $\sigma_i \mapsto \tau_i$, which carries reduced expression on σ_i 's to the corresponding reduced expression on τ_i 's, called *Matsumoto section*. Combining s with the $\mathbb{B}_n$ -action on $V^{\otimes n}$ defined above we set

$$\mathfrak{S}_n = \sum_{\sigma \in \mathbb{S}_n} \rho(s(\sigma)) \in \text{End}(V^{\otimes n}), \quad n \geq 2.$$

This endomorphism is called the *quantum symmetrizer*. For example, $\mathfrak{S}_2 = 1 + c$.

3.2. Nichols algebras. The *Nichols algebra* associated to (V, c) is the quotient

$$\begin{aligned} \mathcal{B}(V) &= T(V) / \bigoplus_{n \geq 2} \ker \mathfrak{S}_n \\ &= \mathbb{k}1 \oplus V \oplus \left(\bigoplus_{n \geq 2} V^{\otimes n} \ker \mathfrak{S}_n \right). \end{aligned}$$

We can check that $\mathcal{B}(V)$ is a $\mathbb{N}_0$ -graded quotient algebra of $T(V)$. We refer to [1] and the references therein for more information on these algebras.

Example 3.2. • If $c(v \otimes w) = w \otimes v$ for all $v, w \in V$, that is c is the usual *flip*, then $\mathcal{B}(V) = S(V)$, the symmetric algebra on V .

• If $c(v \otimes w) = -w \otimes v$ for all $v, w \in V$, then the Nichols algebra is $\mathcal{B}(V) = \Lambda(V)$, the exterior algebra on V .

These are examples of Nichols algebras of diagonal type, that is, those Nichols algebras associated to braided vector spaces of diagonal type. Here the matrix $\mathbf{q}$ is constant. When $\mathbf{q}$ takes arbitrary values, the structure of the associated Nichols algebra $\mathcal{B}_{\mathbf{q}}$ becomes more involved.

Example 3.3. Let (V, c) be a vector space of diagonal type with basis $\{e_i\}_{1 \leq i \leq \theta}$ and $\mathbf{q} = (q_{ij})$, $q_{ij} = q^{a_{ij}}$ for some $q \neq 0, \pm 1$ and a_{ij} as in Example 2.2. For each i we set $\text{ad}_c e_i : T(V) \rightarrow T(V)$ as the linear map

$$\begin{aligned} \text{ad}_c e_i(e_{i_1} \cdots e_{i_k}) \\ = e_i e_{i_1} \cdots e_{i_k} - q_{ii_1} \cdots q_{ii_k} e_{i_1} \cdots e_{i_k} e_i, \\ 1 \leq i, i_1, \dots, i_k \leq \theta. \end{aligned}$$

Now we define recursively $e_{ii} := e_i$, and for $i < j$, $e_{ij} := (\text{ad}_c e_i) \cdots (\text{ad}_c e_{j-1})(e_j)$.

• If q is not a root of unity, then the Nichols algebra is the quotient

$$\mathcal{B}_{\mathbf{q}} = T(V) / \langle (\text{ad}_c e_i)^{a_{ij}} e_j, i \neq j \rangle,$$

and the set

$$\{e_{11}^{n_{11}} e_{12}^{n_{12}} \cdots e_{1\theta}^{n_{1\theta}} e_{22}^{n_{22}} \cdots e_{2\theta}^{n_{2\theta}} \cdots e_{\theta\theta}^{n_{\theta\theta}} \mid n_{ij} \in \mathbb{N}_0\}$$

is a basis of $\mathcal{B}_{\mathbf{q}}$.

• If q is a root of unity of order $N > 1$, then

$$\mathcal{B}_{\mathbf{q}} = T(V) / \langle (\text{ad}_c e_i)^{a_{ij}} e_j, i \neq j; e_{ij}^N, i \leq j \rangle,$$

and the set

$$\{e_{11}^{n_{11}} e_{12}^{n_{12}} \cdots e_{1\theta}^{n_{1\theta}} e_{22}^{n_{22}} \cdots e_{2\theta}^{n_{2\theta}} \cdots e_{\theta\theta}^{n_{\theta\theta}} \mid 0 \leq n_{ij} < N\}$$

is a basis of $\mathcal{B}_{\mathbf{q}}$.

These examples correspond to $U_q^+(\mathfrak{sl}(\theta+1))$ and $\mathfrak{u}_q^+(\mathfrak{sl}(\theta+1))$, the positive parts of the quantized enveloping algebra and the small quantum group of $\mathfrak{sl}(\theta+1)$, respectively.

3.3. PBW bases and root systems of Nichols algebras. The examples above suggest that we can find a basis *à la* PBW, with restricted powers, as we will recall now. Fix (V, c) a braided vector space of diagonal type with matrix $\mathbf{q} = (q_{ij})$; as before, $\mathcal{B}_{\mathbf{q}}$ is the associated Nichols algebra.

- (i) The Nichols algebra $\mathcal{B}_{\mathbf{q}}$ is $\mathbb{N}_0^\theta$ -graded, with each x_i in degree α_i . Here α_i is the θ -uple with 1 in the i -th entry and 0 otherwise.
- (ii) For each $\beta = (b_1, \dots, b_\theta) \in \mathbb{N}_0^\theta$ set

$$q_\beta := \prod_{i,j=1}^{\theta} q_{ij}^{b_i b_j},$$

$$N(\beta) := \begin{cases} \text{ord } q_\beta, & \text{if } q_\beta \text{ is a root of unity,} \\ \infty, & \text{otherwise.} \end{cases}$$

By [9] there exists a restricted PBW basis whose set of *letters* $L \subset \mathcal{B}_q$ is made of homogeneous elements, L has a total order $<$ and

$$\ell_1^{a_1} \cdots \ell_k^{a_k}, \quad \ell_1 < \cdots < \ell_k \in L, \quad 0 < a_i < N(\deg(\ell_i)),$$

is a basis of $\mathcal{B}_q$.

(iii) One can prove that the set

$$\Delta_+^q := \{\deg \ell : \ell \in L\} \subseteq \mathbb{N}_0^\theta$$

does not depend on the chosen PBW basis. Δ_+^q is the *set of positive roots* of q , and $\Delta^q := \Delta_+^q \cup (-\Delta_+^q)$ is the root system.

Remark 3.4. (I) Δ^q is a (generalized) root system in the sense of [7]: there exist *reflections* s_i , but they can take Δ^q to $\Delta^{q'}$ for some $q' \neq q$. This phenomenon was also found for contragredient Lie superalgebras.

(II) Δ^q is the set of degrees of the positive and negative parts of a PBW basis of a Hopf algebra $D(\mathcal{B}_q)$ associated to $\mathcal{B}_q$ which admits a triangular decomposition

$$D(\mathcal{B}_q) \simeq \mathcal{B}_q \otimes \mathbb{C}\mathbb{Z}^\theta \otimes \mathcal{B}_q^*.$$

(III) The matrix q is described by a (*generalized*) *Dynkin diagram*: a certain graph with θ vertices and labels on the edges. There exists a classification of all Nichols algebras $\mathcal{B}_q$ of diagonal type such that $|\Delta^q| < \infty$ in terms of Dynkin diagram, see [6].

3.4. Gelfand-Kirillov dimension and root systems. In the commutative world, we usually study infinite dimensional algebras which are finitely generated and do not grow so much, e.g. by looking at algebras of finite Krull dimension.

Here we want to study infinite dimensional non-commutative algebras A . We still ask for finite generation, so there exists a finite-dimensional subspace V such that $A = \cup_{n \geq 0} V^n$, where $V^n = \{v_1 \cdots v_n | v_i \in V\}$. Set $A_{V,n} = \sum_{0 \leq j \leq n} V^j$. The

Gelfand-Kirillov dimension of A is defined as

$$\text{GK-dim } A := \overline{\lim}_{n \rightarrow \infty} \log_n \dim A_{V,n}.$$

This number does not depend on V . When A is commutative, $\text{GK-dim } A$ is the Krull dimension of A . We refer to [10] for more information on this notion.

Example 3.5. • If $\dim A < \infty$, then $\text{GK-dim } A = 0$. In particular, $\text{GK-dim } \Lambda(V) = 0$ for any finite-dimensional vector space V .

- If $1 < \dim V < \infty$, then $\text{GK-dim } S(V) = \dim V$, $\text{GK-dim } T(V) = \infty$.
- If $\mathfrak{g}$ is a Lie superalgebra, then $\text{GK-dim } U(\mathfrak{g}) = \dim \mathfrak{g}_0$.
- For the algebra $U_q^+(\mathfrak{sl}(\theta + 1))$ in Example 3.3, $\text{GK-dim } U_q^+(\mathfrak{sl}(\theta + 1)) = \binom{\theta+1}{2}$.

One may wonder if there exists a relation between $\text{GK-dim } \mathcal{B}_q$ and Δ^q :

Conjecture 3.6 ([2]). $\text{GK-dim } \mathcal{B}_q < \infty \iff |\Delta^q| < \infty$.

In other words, a Nichols algebra of diagonal type has finite GK-dim if and only if q appears in the list $\mathcal{L}$ of [6]. This conjecture was supported by the evidence in some cases, see e.g. [3]; for example, the case $\theta = 2$ is treated there. Moreover, the implication ($\Leftarrow$) holds by a general argument on filtered algebras, using a filtration induced by the PBW basis. A positive answer to this conjecture was recently obtained in the general case by proving the other implication. More precisely:

Theorem 3.7 ([4]). *If $\text{GK-dim } \mathcal{B}_q < \infty$, then $q \in \mathcal{L}$.*

We finish this exposition with some key ideas behind the proof in [4]:

- As in [3] one may take Nichols algebras $\mathcal{B}_p$ obtained as subquotients of the initial Nichols algebra $\mathcal{B}_q$. Thus $\text{GK-dim } \mathcal{B}_p \leq \text{GK-dim } \mathcal{B}_q$.
- As in [4] we can take an hyperplane $H \subseteq \mathbb{Z}^\theta$ and obtain an appropriate subquotient such that the Nichols algebra $\mathcal{B}_p$ as above has root system $\Delta^q \cap H$. Hence we may argue recursively.

- We use the software GAP [5] in two directions. First, to construct a list $\overline{\mathcal{L}}$ of *candidates* of Dynkin diagrams with θ vertices, obtained by glueing Dynkin diagrams with $\theta - 1$ vertices, which contains properly $\mathcal{L}$. Second, to apply criteria for appropriate hyperplanes in order to discard all the diagrams in $\overline{\mathcal{L}} - \mathcal{L}$.

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ITERATED DISCRIMINANTS AND SINGULAR INTERSECTIONS OF HYPERSURFACES

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ABSTRACT. It is well known that two generic quadric surfaces intersect in a nonsingular quartic space curve, but when the intersection is not transverse this curve may degenerate to a finite number of different possible types of singular curves. In the nice paper [9], the authors formulate a way of computing the condition for a degenerate intersection in this case, which refines in the real case and with an algorithmic point of view a classical treatise by Bromwich [1]. Independently, Schläfli [18] studied the degenerate intersection of two hypersurfaces described by multilinear equations.

I will describe in this note a joint work with Sandra di Rocco and Ralph Morrison. We presented in [6] a general framework of iterated sparse discriminants to characterize the singular intersection of hypersurfaces with a given monomial support A , which generalizes both previous situations. We studied the connection of iterated discriminants with the notion of mixed discriminant [3, 7] and the singularities of the sparse discriminant [10] associated to A .

1. INTRODUCTION

If we look at the three parabolas in Figure 1 we see that in general two different parabolas have two points of intersection, but that if the intersection is not transverse (that is, if they have the same tangent line), it can consist of a single point.

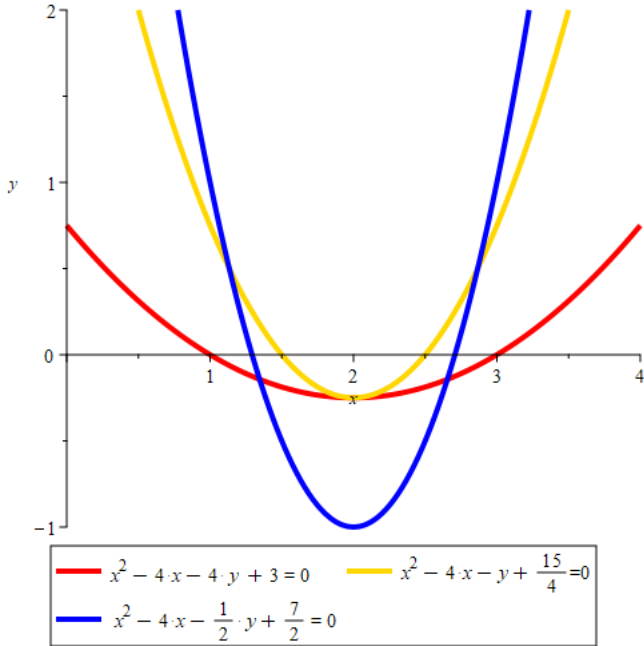


FIGURE 1. The two upper parabolas don't have a transverse intersection

In three dimensions, the intersection of two quadric surfaces is in general a smooth space curve of degree 4. The possible singular intersections over the complex numbers were classically studied in [1, 17, 18] and more recently in [9] in the context of Computer Aided Geometry Design for real intersections. This is what they did to describe the singular intersections $(f_0 = 0) \cap (f_1 = 0)$ of two quadric surfaces. Write $f_i(x_1, x_2, x_3) = XM_iX^t$, where $M_i \in \mathbb{R}^{4 \times 4}$ is symmetric and $X = (1, x_1, x_2, x_3)$. The entries of each matrix M_i are simply coefficients from f_0, f_1 , in some cases scaled by $\frac{1}{2}$. Consider the pencil of quadrics $f_\lambda = (1 - \lambda)f_0 + \lambda f_1$ depending on a parameter λ , and compute the determinant

$$d_f(\lambda) = \det((1 - \lambda)M_0 + \lambda M_1).$$

This is a polynomial in λ of degree 4. Then, they detected for which quadrics f_0, f_1 (or equivalently, for which matrices M_0, M_1) the polynomial d_f has a multiple root. The classification is done via the so called Segre characteristic, obtained from the Jordan normal form.

Schläfli used a similar method in [18] for the study of hyperdeterminants, that is, in the case of

multilinear polynomials. Also in geometric modeling, a similar method which goes back to [13] and [5] was used to classify Darboux cyclides. These are surfaces in 3 space defined by polynomials of the form: $c(x^2 + y^2 + z^2)^2 + (x^2 + y^2 + z^2)L(x, y, z) + Q(x, y, z)$, where c is a constant, L a linear form and Q is a quadric. These surfaces are the projection to 3 space of the intersection of two quadrics in 4 space (one is the unit sphere) [4]. They were more recently studied by [20] with a similar approach.

The leading question of our work [6] is why this works, what is behind this method and how to extend it. The keyword is: *discriminants*. They were computing *mixed discriminants* [3, 7, 8, 10] using *iterated discriminants*, a notion previously considered in [2, 14] that we define in general. In fact, we proved that their method essentially only works for quadrics and we explain which other factors occur in iterated discriminants. We show that these extra terms arise from the singular locus of the sparse discriminant [10, 19].

2. FROM UNIVARIATE DISCRIMINANTS TO A DISCRIMINANTS

We all know that a degree 2 univariate polynomial $f = ax^2 + bx + c$, $a \neq 0$, has a double root if and only if the discriminant $D_{\{0,1,2\}}(f) = b^2 - 4ac = 0$. This is a polynomial of degree 2 in the coefficients of f . Similarly, for any n , there exists an irreducible *discriminant* polynomial $D_{\{0,1,\dots,n\}}(f) \in \mathbb{Z}[c_0, \dots, c_n]$ of degree $2(n-1)$ (defined up to sign), which vanishes on the coefficients of a polynomial $f = c_n x^n + \dots + c_0$ of degree n whenever not all roots of f are simple.

Let's look at three ways of computing $D_{\{0,1,2\}}(f)$. A first way is to compute the resultant $\text{Res}(f, df/dx)$ of f and its derivative using Sylvester formula and then divide by the leading

coefficient a :

$$\begin{aligned} \text{Res}(f, df/dx) &= \det \begin{pmatrix} a & b & c \\ 2a & b & 0 \\ 0 & 2a & b \end{pmatrix} \\ &= -a(b^2 - 4ac). \end{aligned}$$

A second way is to homogenize the input polynomial f with another variable y . We then consider the polynomial $F = ax^2 + bxy + cy^2$. Note that by the Euler relation we have that $2F = xF_x + yF_y$, with $F_x = 2ax + by$, $F_y = bx + 2cy$. So, we compute the condition under which the two partial derivatives have a nonzero point of intersection:

$$\det \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} = b^2 - 4ac = 0.$$

A third way is to solve $x = -\frac{b}{2a}$ from $df/dt = 0$, replace it in $f = 0$ and clear denominators. Or more generally, via elimination of variables using an elimination order in a Computer Algebra System.

If our input is a quadric f in n variables, we try similarly to compute a polynomial in its coefficients that vanishes whenever there is a point where f and all the partials vanish (at a point with nonzero coordinates). This polynomial is again irreducible, has integer coefficients, and it is unique up to sign if we ask that its content is equal to 1. Homogenize f as $F(x_0, x_1, \dots, x_n)$. Then,

$$F(x_0, \dots, x_n) = (x_0, \dots, x_n)M(x_0, \dots, x_n)^t,$$

If the hypersurface ($f = 0$) has a singular point, there exists a nontrivial common solution of $\partial F/\partial x_0 = \dots = \partial F/\partial x_n = 0$. This happens precisely when $\det(M) = 0$. The discriminant of f equals $\pm \det(M)$.

The far-reaching general definition of A -discriminants was introduced by Gelfand, Kapranov and Zelevinsky [10], as a by-product of their study of singularities of multivariate hypergeometric functions. They are also called *sparse discriminants*. Fix $A \subset \mathbb{Z}^n$ finite and consider polynomials with support A

$$f(x) = \sum_{a \in A} c_a x^a, x = (x_1, \dots, x_n).$$

The A -discriminant is the unique (up to sign) irreducible polynomial D_A with integer coefficients and content 1 in the unknowns $\mathbf{c} = (c_a : a \in A)$ which vanishes whenever f has a *multiple root* in the torus $(\mathbb{C}^*)^n$: $(D_A = 0) = \overline{C}$, where

$$C = \{\mathbf{c} : \exists x^* \in (\mathbb{C}^*)^n \text{ such that}$$

$$f(x^*) = \frac{\partial f}{\partial x_1}(x^*) = \cdots = \frac{\partial f}{\partial x_n}(x^*) = 0\},$$

assuming the right hand side defines a hypersurface. This happens for *generic* A and we say that A is *non-defective*. In this case, $(D_A = 0)$ is an equation of the dual variety $X_A^\vee$ of the projective toric variety X_A rationally parametrized sending $t \in (\mathbb{C}^*)^n$ to the projective point with coordinates $t^a, a \in A$ (in some order). This widely generalizes the previous univariate and quadratic examples. There are several ways to compute A -discriminants extending the three approaches we mentioned in the case of univariate polynomials, but the general theory is much more complicated and gave rise to numerous publications that we cannot cite here.

Mixed discriminants, as well as resultants and hyperdeterminants, are special cases of A -discriminants via the Cayley trick. Again, fix a finite set $A \subset \mathbb{Z}^n$ of exponents and let $f_0, \dots, f_r$ be Laurent polynomials with support A and coefficients in an algebraically closed field of characteristic 0:

$$f_i = \sum_{a \in A} c_a^i x_1^{a_1} \cdots x_n^{a_n}.$$

The mixed discriminant $MD_{A, \dots, A}(f_0, \dots, f_r)$ is the irreducible polynomial in the coefficients of $f_0, \dots, f_r$ which vanishes if there is a non-transverse intersection. When this is a codimension 1 condition, by a result by Katz [12] it is enough to consider *non-degenerate* non-transverse intersections. Note that the case $r = 0$ corresponds to A -discriminants.

Example 2.1. Assume that $A = \{(0, 0), (1, 0), (0, 1), (1, 1)\}$ corresponds to the vertices of the unit square in the plane and take two polynomials with support A : $f_0 = a_{00} + a_{10}x + a_{01}y + a_{11}xy$, $f_1 = b_{00} + b_{10}x + b_{01}y + b_{11}xy$. Then, the associated mixed discriminant

$MD_{A,A}(f_0, f_1) = MD_{A,A}(a, b)$ is the *hyperdeterminant* of format $2 \times 2 \times 2$:

$$(2.1) \quad \begin{aligned} & a_{00}^2 b_{11}^2 - 2a_{00}a_{01}b_{10}b_{11} - 2a_{00}a_{10}b_{01}b_{11} \\ & - 2a_{00}a_{11}b_{00}b_{11} + 4a_{00}a_{11}b_{01}b_{10} + a_{01}^2 b_{10}^2 \\ & + 4a_{01}a_{10}b_{00}b_{11} - 2a_{01}a_{10}b_{01}b_{10} - 2a_{01}a_{11}b_{00}b_{10} \\ & + a_{10}^2 b_{01}^2 - 2a_{10}a_{11}b_{00}b_{01} + a_{11}^2 b_{00}^2. \end{aligned}$$

It holds that $MD_{A,A}(a, b) = 0$ whenever the two hyperbolas $(f_0 = 0)$ and $(f_1 = 0)$ are tangent. As we mentioned, by classical work of Schläfli [18] $MD_{A,A}$ can be computed via a pencil.

3. ITERATED DISCRIMINANTS

Assume A is non-defective, $\deg(D_A) = \delta > 0$, and set $r \leq \dim(X_A)$. Let $f_0, \dots, f_r$ with support A , with respective vectors of coefficients $c^0, \dots, c^r$ and $\lambda = (\lambda_0, \dots, \lambda_r)$. Denote $f_\lambda = \lambda_0 f_0 + \cdots + \lambda_r f_r$. Think of λ as parameters and consider $D_A(f_\lambda)$. This is a polynomial in $\mathbb{Z}[c^0, \dots, c^r][\lambda]$ of degree δ in λ . Let D_δ be the sparse discriminant of the family of polynomials of degree δ in $r + 1$ homogeneous variables. We define the iterated discriminant $ID_{r,A}$ by:

$$ID_{r,A}(f_0, \dots, f_r) = D_\delta(D_A(f_\lambda)) \in \mathbb{Z}[c^0, \dots, c^r].$$

In case $r > \dim(X_A) = \dim_{\text{affine}}(A)$, we define $ID_{r,A}(f_0, \dots, f_r) = 0$.

In the classical examples, they were computing a mixed discriminant using the iterated discriminant, in the case $r = 1$ of two quadric surfaces f_0, f_1 , represented by 4×4 symmetric matrices M_0 and M_1 . This gives a compact way of computing $MD_{A,A}(f_0, f_1)$. Expanding this expression in terms of the coefficients of M_1, M_2 is beyond the capabilities of Macaulay2 [11], as a general polynomial of bidegree $(12, 12)$ has more than $4 \cdot 10^{11}$ monomials!

We present two examples that are basic instances of our main results.

Example 3.1. Consider the two dimensional configuration corresponding to the first *Hirzebruch surface* $\mathbb{F}_1$: $A = \{(0, 0), (1, 0), (2, 0), (0, 1), (1, 1)\}$. Given the family of polynomials of the form

$a_0 + a_1x + a_2x^2 + y(b_0 + b_1x)$, the associated A -discriminant coincides with the *resultant* of the two univariate polynomials $a_0 + a_1x + a_2x^2$ and $b_0 + b_1x$ and is thus equal to the degree 3 polynomial $D_A(f) = a_0b_1^2 - a_1b_0b_1 + a_2b_0^2$. The ideal I defining the singular locus S of $(D_A = 0)$ is generated by $b_0^2, b_0b_1, b_1^2, -a_1b_1 + 2a_2b_0, 2a_0b_1 - a_1b_0$. Its radical is generated by b_0, b_1 (and $\text{rad}(I)^2 \subset I$).

The mixed discriminant $MD_{A,A}(f_0, f_1)$ has degree 8, while the iterated discriminant $ID_{1,A}(f_0, f_1)$ has degree 12. There is another factor Ch_S^2 of degree $2 \times 2 = 4$, arising from the *Chow form* of S . If f_0 has coefficients (a, b) and f_1 has coefficients (A, B) , it equals: $Ch_S((a, b), (A, B)) = B_0b_1 - B_1b_0$. We recall that Ch_S is the defining polynomial of the irreducible variety in the Grassmannian of lines that intersect the codimension 2 variety S .

Example 3.2. Let $A = \{0, 1, 2, 3\}$, $r = 1$. Given two cubic univariate polynomials p_0, p_1 , their mixed discriminant equals the discriminant of the Cayley configuration

$$C = \{(0, 0), (0, 1), (0, 2), (0, 3), \\ (1, 0), (1, 1), (1, 2), (1, 3)\}$$

at the polynomial $p_0 + \lambda p_1$ in one more variable λ . In fact, $D_C(p_0 + \lambda p_1)$ equals the resultant $\text{Res}_{3,3}(p_0, p_1)$, which has degree 6 in the coefficients of p_0, p_1 . Since the discriminant D_A of a cubic univariate polynomial has degree 4, the iterated discriminant $ID_{1,A}$ instead has degree $2 \cdot 4 \cdot 3 = 24$. It has another irreducible factor of degree 6 raised to the third power, which is given by the Chow form of the singular locus of $D_A = 0$ corresponding to degree 3 polynomials with a *triple root* (the first instance of a degenerate multiple root).

4. OVERVIEW OF OUR RESULTS IN [6]

We show that in the non-defective case $MD_{r,A}$ is always an irreducible factor of $ID_{r,A}$. Note that even if they are not equal, this result implies that if $ID_{r,A}(f_0, \dots, f_r) \neq 0$, we get a *certificate* that the intersection $(f_0 = \dots = f_r)$ is smooth.

When A is non-defective, we denote by $\text{sing}(X_A^\nu)$ the subscheme of the dual variety X_A^ν defined by the ideal generated by the partial derivatives of D_A . As the previous examples suggest, we show that $ID_{r,A}$ can have other irreducible factors given by the Chow forms of the higher dimensional irreducible components of the singular locus. Here is the precise statement. Assume $A \subset \mathbb{Z}^n$ is a finite non-defective configuration and let $0 \leq r \leq \dim(X_A)$. Denote by c the codimension of $\text{sing}(X_A^\nu)$ in the dual variety $X_A^\nu = (D_A = 0)$.

Theorem 4.1 (Theorem 4.4 in [6]). *The mixed discriminant $MD_{r,A}$ divides $ID_{r,A}$. Moreover,*

- (1) *If $c > r$, then $ID_{r,A} = MD_{r,A}$.*
- (2) *If $c = r$, $ID_{r,A} = MD_{r,A} \prod_{i=1}^{\ell} Ch_{Y_i}^{\mu_i}$, where $Y_1, \dots, Y_\ell$ are the irreducible components of $\text{sing}(X_A^\nu)$ of codimension r , with respective multiplicities $\mu_i \geq 2$.*
- (3) *If $c < r$, then $ID_{r,A} = 0$.*

We conjecture that $\mu_k = 2$ if Y_k is a component of the *double point locus* of those points for which there are two different non-degenerate multiple roots, while $\mu_k = 3$ when Y_k is a component of the *cusp locus* of those p for which there is a *degenerate* multiple root.

One of the main tools in the proof of Theorem 4.1 is the property known as *biduality* [10, Chapter 1] that in case $(f = 0)$ has a single singularity of multiplicity 2 at a point p in the torus, we can recover p from the gradient of the A -discriminant evaluated at (the coefficients of) f .

The next question is for which configurations A the mixed discriminant and the iterated discriminant coincide. We need to determine when the codimension c of the singular locus of the dual variety X_A^ν is bigger than 1. This is the case for two quadrics in any number of variables [16]. We investigate when these other irreducible factors do not occur in two contexts. The general answer is related to the difficult problem of determining the codimension and the components of $\text{sing}(X_A^\nu)$, which are in general unknown. They have been only fully determined in the univariate case and

for multilinear polynomials. Weyman and Zelevinsky proved in [19] that it works for bilinear forms in any number of variables (degree 2) or trilinear forms in 3 variables (degree 3) and that these last two cases are the only ones among hyperdeterminants, the equations of the duals of Segre varieties. They also do a detailed study of the singularities of hyperdeterminants.

Theorem 4.2 (Theorem 5.6 in [6]). *The only cases of Segre-Veronese varieties $X_A = \mathbb{P}^{k_1}(d_1) \times \cdots \times \mathbb{P}^{k_\ell}(d_\ell)$ with all $d_i \geq 2$, for which a mixed discriminant can be computed as an iterated discriminant is when $X_A = \mathbb{P}^n(2)$ for any $n \geq 1$ (i.e. in the known case of two quadrics in any dimension).*

When $c \geq r$, we show that the degree of the iterated discriminant equals

$$\deg(ID_{r,A}) = (r+1)\delta(\delta-1)^r,$$

where $\delta = \deg(X_A^\vee)$. Our proof of Theorem 4.2 is combinatorially quite complicated, comparing degrees. We conjecture that there are no other cases if some but not all $d_i = 1$.

We also prove the following result for smooth lattice polygons. Recall that a polygon P in $\mathbb{R}^2$ is said to be a *lattice* polygon when it has integer vertices. A lattice polygon is *smooth* when for any vertex $v \in P$, with adjacent points v_1, v_2 , one in each of the two concurring edges of P at v , the integer vectors $v_1 - v, v_2 - v$ are a $\mathbb{Z}$ basis of $\mathbb{Z}^2$.

Theorem 4.3 (Theorem 6.3 in [6]). *Given a configuration A of lattice points of a smooth lattice polygon in the plane, the only cases when $(D_A = 0)$ has singularities of codimension $c > 1$ are the 2-triangle $\text{conv}((0,0), (2,0), (0,2))$, $X_A = \mathbb{P}^2(2)$, and the square $\text{conv}((0,0), (1,0), (0,1), (1,1))$, $X_A = \mathbb{P}^1 \times \mathbb{P}^1$. That is, these two are the only cases in which the mixed discriminant $MD_{1,A}$ is equal to the iterated discriminant $ID_{1,A}$.*

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NONLINEAR PROGRAMMING AND AUGMENTED LAGRANGIANS

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ABSTRACT. The classical augmented Lagrangian method is a numerically efficient tool for solving nonlinear programming problems. The augmented Lagrangian function was introduced to improve the stability of a numerical method. Subsequent studies used this function to reinterpret Lagrange multipliers, to introduce constraint qualifications and to characterize second-order optimality conditions. We summarize some properties of the classical and sharp augmented Lagrangian function and its relation with the numerical method.

1. INTRODUCTION

Consider the mathematical programming problem

$$(1.1) \quad \begin{aligned} & \text{minimize} && f(x) \\ & \text{subject to} && g_i(x) = 0 \quad \forall i \in \mathcal{E}, \\ & && g_i(x) \leq 0 \quad \forall i \in \mathcal{I}, \\ & && x \in \Omega, \end{aligned}$$

where $f : \mathbb{R}^n \mapsto \mathbb{R}$ and $g : \mathbb{R}^n \mapsto \mathbb{R}^m$, with $m = |\mathcal{E}| + |\mathcal{I}|$, are twice continuously differentiable and Ω is a polyhedral set. Let $Q = \{\eta \mid \eta_i \in \mathbb{R}, i \in \mathcal{E}; \eta_i \geq 0, i \in \mathcal{I}\}$. Then equality and inequality constraints can be written with the polar cone $Q^\circ = \{\xi \mid \xi_i = 0, i \in \mathcal{E}; \xi_i \leq 0, i \in \mathcal{I}\}$, where for an inner product $\langle \cdot, \cdot \rangle$ the polar of a cone K is $K^\circ = \{\mu \mid \langle \mu, v \rangle \leq 0 \quad \forall v \in K\}$.

Since the feasible set of problem (1.1) is $D = \{x \in \Omega \mid g(x) \in Q^\circ\}$, we have that if $\bar{x}$ is a local solution of problem (1.1) then

$$\langle \nabla f(\bar{x}), u \rangle \geq 0, \quad \forall u \in \mathcal{T}_D(\bar{x}),$$

where

$$\mathcal{T}_C(\bar{z}) = \{u \mid \exists \tau_k \rightarrow 0^+, u^k \rightarrow u \text{ s.t. } \bar{z} + \tau_k u^k \in C\}$$

is the tangent cone to C at $\bar{z} \in C$. This first order necessary optimality condition is equivalent to the generalized equation

$$0 \in \nabla f(\bar{x}) + \mathcal{N}_D(\bar{x}),$$

where $\mathcal{N}_C(\bar{z}) = \mathcal{T}_C(\bar{z})^\circ$ is the normal cone to C at $\bar{z} \in C$. It can be seen that

$$\mathcal{T}_D(\bar{x}) \subset \{u \in \mathcal{T}_\Omega(\bar{x}) \mid g'(\bar{x})u \in \mathcal{T}_{Q^\circ}(g(\bar{x}))\},$$

that is, the geometric cone is a subset of the analytic cone. By polarity we have

$$\mathcal{N}_D(\bar{x}) \supset \{g'(\bar{x})^T \mu + \nu \mid \nu \in \mathcal{N}_\Omega(\bar{x}), \mu \in \mathcal{N}_{Q^\circ}(g(\bar{x}))\}.$$

If those cones are equal, it is said that the Guignard constraint qualification (GCQ) holds at $\bar{x}$. Under GQC we can guarantee the existence of $\mu \in \mathbb{R}^m$ such that

$$(1.2) \quad 0 \in \nabla_x L(\bar{x}, \mu) + \mathcal{N}_\Omega(\bar{x}), \quad \mu \in \mathcal{N}_{Q^\circ}(g(\bar{x})),$$

where $L(x, \mu) = f(x) + \langle \mu, g(x) \rangle$ is the Lagrangian function for problem (1.1). The generalized equation (1.2) is called the Karush-Kuhn-Tucker system and those μ satisfying (1.2) are the elements of the Lagrange multipliers set, denoted by $\mathcal{M}(\bar{x})$.

2. PRIMAL AND DUAL PROBLEM

Since Q is a closed convex cone, it can be seen that $\mu \in \mathcal{N}_{Q^\circ}(g(\bar{x}))$ iff $g(\bar{x}) \in \mathcal{N}_Q(\mu)$. Using that $g(\bar{x}) = \nabla_\mu L(\bar{x}, \mu)$, then $(\bar{x}, \bar{\mu})$ satisfies (1.2) iff

$$\begin{aligned} 0 & \in \nabla_x L(\bar{x}, \bar{\mu}) + \mathcal{N}_\Omega(\bar{x}), \\ 0 & \in -\nabla_\mu L(\bar{x}, \bar{\mu}) + \mathcal{N}_Q(\bar{\mu}). \end{aligned}$$

Note that the first generalized equation is associated to minimize $L(x, \bar{\mu})$ and the second one to maximize $L(\bar{x}, \mu)$. By definition of L we have that

$$\sup_{\mu \in Q} L(x, \mu) = \begin{cases} f(x), & \text{if } g(x) \in Q^\circ, \\ \infty, & \text{otherwise.} \end{cases}$$

Hence, if $\Omega_\varepsilon := \Omega \cap B(\bar{x}, \varepsilon)$, the (local) primal problem is minimize $f(x)$, which is equivalent to

$$\text{minimize} \left\{ \sup_{x \in \Omega_\varepsilon} L(x, \mu) \right\},$$

and its optimal value is

$$f(\bar{x}) = \inf_{x \in \Omega_\varepsilon} \left\{ \sup_{\mu \in Q} L(x, \mu) \right\}.$$

Analogously, defining $\varphi(\mu) = \inf_{x \in \Omega_\varepsilon} L(x, \mu)$, the dual problem is

$$\text{maximize}_{\mu \in Q} \varphi(\mu),$$

and its optimal value is $\vartheta = \sup_{\mu \in Q} \{\inf_{x \in \Omega_\varepsilon} L(x, \mu)\}$. Note that $\vartheta \leq f(\bar{x})$. If $\vartheta < f(\bar{x})$ the problem (1.1) has a duality gap at $\bar{x}$.

For analyzing the duality gap, define the value function as

$$v(p) = \inf_{x \in D(p)} f(x),$$

where $D(p) = \{x \in \Omega_\varepsilon \mid g(x) - p \in Q^\circ\}$. Then $v(0) = f(\bar{x})$. On the other hand, if $x \in \Omega_\varepsilon$ and $\mu \in Q$ we have

$$\begin{aligned} \inf_{\{p \mid g(x) - p \in Q^\circ\}} \{f(x) + \langle \mu, p \rangle\} &= \inf_{q \in Q^\circ} \{L(x, \mu) - \langle \mu, q \rangle\} \\ &= L(x, \mu). \end{aligned}$$

Thus, for $\mu \in Q$,

$$\begin{aligned} \varphi(\mu) &= \inf_{x \in \Omega_\varepsilon} L(x, \mu) = \inf_p \inf_{x \in D(p)} \{f(x) + \langle \mu, p \rangle\} \\ &= \inf_p \{v(p) + \langle \mu, p \rangle\}. \end{aligned}$$

Hence, we have that $\bar{x}$ and $\bar{\mu}$ solve the primal and dual problem, respectively, iff $\varphi(\bar{\mu}) = f(\bar{x})$, which is equivalent to

$$v(p) \geq v(0) - \langle \bar{\mu}, p \rangle, \quad \forall p$$

For a graphical interpretation, recall that the Fenchel biconjugate function v^{**} is the minorant of v obtained by convexification of v , [5, Chapter 11.A], and for this particular case it holds that

$$\begin{aligned} v^{**}(0) &= \sup_{\mu} \left\{ \inf_p \{v(p) + \langle \mu, p \rangle\} \right\} = \sup_{\mu \in Q} \varphi(\mu) \\ &= \vartheta \leq f(\bar{x}) = v(0). \end{aligned}$$

Hence, if $v^{**}(0) = v(0)$ no duality gap exists at $\bar{x}$. See Figure 1.

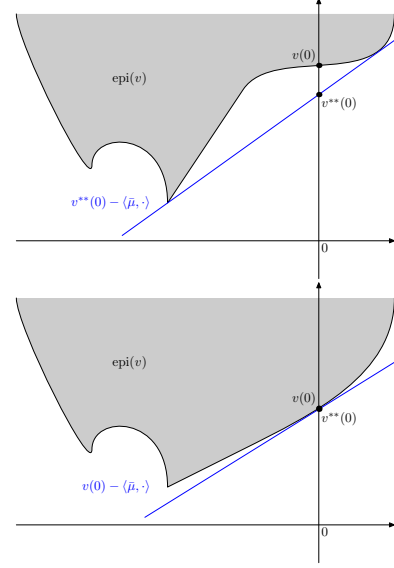


FIGURE 1. Duality gap representation in terms of the value function.

3. CLASSICAL AUGMENTED LAGRANGIAN

The classical augmented Lagrangian function is defined by

$$\bar{L}(x; \mu, r) = f(x) + \frac{1}{2r} (\|\Pi_Q(\mu + rg(x))\|^2 - \|\mu\|^2),$$

where Π_Q is the orthogonal projection onto Q . The augmented Lagrangian method generates sequences such that

(1) x^{k+1} is a (global) solution of

$$\text{minimize}_{x \in \Omega} \bar{L}(x; \mu^k, r_k),$$

(2) $\mu^{k+1} = \Pi_Q(\mu^k + r_k g(x^{k+1}))$,

(3) $r_{k+1} \geq r_k$.

For practical implementations an inexact version is used. It defines x^{k+1} s.t

$$\|x^{k+1} - \Pi_Q(x^{k+1} - \nabla_x \bar{L}(x^{k+1}; \mu^k, r_k))\| \leq \varepsilon_k,$$

for an exogenous sequence $\varepsilon_k \rightarrow 0^+$. For a theoretical interpretation, we have that

$$\bar{L}(x; \mu, r) = \inf_{\{p \mid g(x) - p \in Q^\circ\}} \left\{ f(x) + \langle \mu, p \rangle + \frac{r}{2} \|p\|^2 \right\}.$$

See [5, Example 11.57]. Then, in terms of the value function we obtain

$$\begin{aligned} \inf_{x \in \Omega_\varepsilon} \bar{L}(x; \mu, r) &= \inf_p \inf_{x \in D(p)} \left\{ f(x) + \langle \mu, p \rangle + \frac{r}{2} \|p\|^2 \right\} \\ &= \inf_p \left\{ v(p) + \langle \mu, p \rangle + \frac{r}{2} \|p\|^2 \right\}. \end{aligned}$$

Hence, if $f(\bar{x}) \leq \bar{L}(x; \bar{\mu}, r)$ for all $x \in \Omega_\varepsilon$ we have

$$v(p) \geq v(0) - \langle \bar{\mu}, p \rangle - \frac{r}{2} \|p\|^2, \quad \forall p.$$

As shown in Figure 2, in this case the epigraph of v is supported by a paraboloid. The former holds under the second-order sufficient optimality condition (SOSC):

$$\begin{aligned} \exists \bar{\mu} \in \mathcal{M}(\bar{x}) \text{ s.t. } \forall u \in \mathcal{C}(\bar{x}) \setminus \{0\}, \\ \langle \nabla_{xx}^2 L(\bar{x}, \bar{\mu}) u, u \rangle > 0, \end{aligned}$$

where

$$\mathcal{C}(\bar{x}) = \{u \in \mathcal{T}_\Omega(\bar{x}) \mid g'(\bar{x})u \in \mathcal{T}_{Q^\circ}(g(\bar{x}))\} \cap \nabla f(\bar{x})^\perp$$

is the cone of critical directions. By [1, Theorem 5.1(1), Theorem 5.2(1)], we have that SOSC is equivalent to $\bar{L}(x; \bar{\mu}, r) \geq f(\bar{x}) + \gamma \|x - \bar{x}\|^2$ for all $x \in \Omega_\varepsilon$.

The previous analysis of the augmented Lagrangian function was used to improve the convergence theory of the augmented Lagrangian method. By [3, Theorem 3.4], for any (x^0, μ^0) close enough to $(\bar{x}, \bar{\mu})$ satisfying SOSC, the sequence $\{(x^k, \mu^k)\}$ converges to $(\bar{x}, \hat{\mu})$ with $\hat{\mu} \in \mathcal{M}(\bar{x})$.

4. SHARP AUGMENTED LAGRANGIAN

The sharp augmented Lagrangian function is defined by

$$\tilde{L}(x; \mu, r) = \inf_{\{p \mid g(x) - p \in Q^\circ\}} \{f(x) + \langle \mu, p \rangle + r \|p\|\}.$$

See [5, Example 11.58]. In particular, for $Q^\circ = \{0\}$ we have $\tilde{L}(x; \mu, r) = f(x) + \langle \mu, g(x) \rangle + r \|g(x)\|$. For $\mu = 0$ we have $\tilde{L}(x; \mu, r) = f(x) + r \|\Pi_Q(g(x))\|$. In terms of the value function we obtain that if $f(\bar{x}) \leq \tilde{L}(x; \bar{\mu}, r)$ for all $x \in \Omega_\varepsilon$, then

$$v(p) \geq v(0) - \langle \bar{\mu}, p \rangle - r \|p\|, \quad \forall p.$$

As shown in Figure 2, in this case the epigraph of v is supported by a cone. The former holds under the weak second-order sufficient optimality condition (WSOSC):

$$\begin{aligned} \forall u \in \mathcal{C}(\bar{x}) \setminus \{0\}, \exists \bar{\mu} \in \mathcal{M}(\bar{x}) \text{ s.t.} \\ \langle \nabla_{xx}^2 L(\bar{x}, \bar{\mu}) u, u \rangle > 0. \end{aligned}$$

By [1, Theorem 5.1(2), Theorem 5.2(2)], we have that WSOSC is equivalent to $\tilde{L}(x; \bar{\mu}, r) \geq f(\bar{x}) + \gamma \|x - \bar{x}\|^2$ for all $x \in \Omega_\varepsilon$. Now, if in the augmented Lagrangian method we change $\bar{L}$ by $\tilde{L}$ we obtain a significant improvement, according to [4].

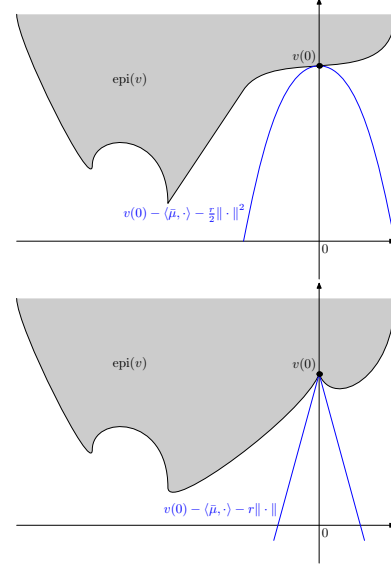


FIGURE 2. Duality gap removed by augmented Lagrangians.

Unfortunately, the function $x \mapsto \tilde{L}(x; \mu, r)$ is nonconvex and nonsmooth, thus there are not practical numerical implementations. Nevertheless, based on [1, Proposition 6.1] we have that

$$\begin{aligned} \tilde{L}(x; \mu, r) &= \inf_{t \geq 0} \left\{ \bar{L}(x; \mu, \frac{r}{t}) + \frac{r}{2} t \right\} \\ &= \inf_{t \geq 0} \{ \tilde{L}(x, t; \mu, r) \}, \end{aligned}$$

where $\tilde{L}(x, t; \mu, r) = \bar{L}(x; \mu, \frac{r}{t}) + \frac{r}{2} t$ if $t > 0$ and $\tilde{L}(x, t; \mu, r) = f(x)$ if $t = 0, g(x) \in Q^\circ$. The domain of $\tilde{L}$ is $\{(x, t) \mid t > 0\} \cup \{(x, 0) \mid g(x) \in Q^\circ\}$, it is lower semicontinuous and for (x, t) with $t > 0$ it is continuously differentiable with

$$\begin{aligned} \nabla_x \tilde{L}(x, t; \mu, r) &= \nabla f(x) + g'(x)^T \Pi_Q \left(\mu + \frac{r}{t} g(x) \right), \\ \frac{\partial \tilde{L}}{\partial t}(x, t; \mu, r) &= \frac{r}{2} - \frac{1}{2r} \left\| \Pi_Q \left(\mu + \frac{r}{t} g(x) \right) - \mu \right\|^2. \end{aligned}$$

Hence, the proposed sharp augmented Lagrangian method generates sequences such that

(1) (x^{k+1}, t_{k+1}) is a (global) solution of

$$\underset{(x,t) \in \Omega \times [0, \infty)}{\text{minimize}} \quad \tilde{L}(x, t; \mu^k, r_k),$$

- (2) $\mu^{k+1} = \Pi_Q \left(\mu^k + \frac{r_k}{t_{k+1}} g(x^{k+1}) \right),$
- (3) $r_{k+1} \geq r_k.$

The sequences generated by this method have the same properties of the generated by the methods proposed in [4]. For practical implementations an inexact version should generate (x^{k+1}, t_{k+1}) with $t_{k+1} > 0$, as showed in [2].

5. CONCLUSIONS AND FUTURE WORK

In this work we showed that the analytical study of augmented Lagrangian functions is a powerful tool to understand the behaviour of numerical methods. This analysis allows us to obtain a convergence theory weakening hypothesis. Also, allows us to proposed modifications of the method to improve its convergence. Partial results about the convergence theory and numerical results of the sharp augmented Lagrangian method were presented in [6] when problem (1.1) has only equality constraints, i.e., $\mathcal{I} = \emptyset$. Numerical results and convergence theory for the general problem is still in process.

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ON POINCARÉ SERIES FOR $SL_2(\mathbb{R})$ AND $SU(2, 1)$

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Let $H = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$ equipped with the hyperbolic metric $\frac{dx dy}{y^2}$. In this case, the geodesics are semicircles and half-lines perpendicular to the x -axis. This geometry gave a counter-example to Euclid's fifth axiom twenty two centuries later (with work of Bolyai (1832), Lobachevskii (1829) and Gauss (1830)). Further, the area of a hyperbolic triangle with interior angles α, β, γ is $A = \pi - (\alpha + \beta + \gamma)$ and the group of isometries is

$$I(H)_0 = \left\{ z \rightarrow g(z) := \frac{az + b}{cz + d}; \right. \\ \left. g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{R}) \right\}$$

The isotropy subgroup of $z = i$ is

$$SL_2(\mathbb{R})_i = \left\{ \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} : 0 \leq \theta \leq 2\pi \right\} \\ \simeq SO(2) := K.$$

Thus, $H \approx G \setminus K$ is a homogeneous riemannian manifold. Let $\Gamma \subset SL_2(\mathbb{R})$ be a discrete subgroup of G such that $\text{vol}(\Gamma \setminus H) < \infty$, i.e. Γ is a lattice in G . Such a subgroup has a fundamental region given by a hyperbolic polygon with finitely many sides. Two standard examples are

$$\Gamma(n) = \left\{ \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) : \gamma \equiv Id \pmod{n} \right\},$$

and the Hecke group

$$\Gamma_0(n) := \left\{ \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \pmod{n} \right\},$$

for each fixed $n \in \mathbb{N}$.

In particular, $\Gamma(1) = SL_2(\mathbb{Z})$ and

$$\Gamma(2) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix}, ad - bc = 1, b, c \in 2\mathbb{Z} \right\}.$$

Definition. Let $k \in \mathbb{Z}$, $k \geq 3$. A Γ -automorphic form of weight k is a holomorphic function

$$f : H \rightarrow \mathbb{C}$$

on H such that

$$f(\gamma z) = (cz + d)^k f(z), \quad \forall z \in H,$$

and such that f is holomorphic at the cusps of Γ .

There is fundamental region for the action of Γ on $H \cup i_\infty$, that consists of a hyperbolic polygon with finitely many sides, and each cusp of Γ is represented by a vertex of this polygon fixed by a unipotent element of Γ .

In the case of the cusp i_∞ , f holomorphic at i_∞ means that

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}$$

the series converging uniformly on compacts in H . This is called the Fourier expansion of f at i_∞ . At the other cusps of Γ there is a similar expansion. Then, when the coefficient $a_0 = 0$ for each cusp of Γ , f is said to be a *cusp form*.

Remark. In we form $H^* = H \cup \{\text{cusps of } \Gamma\}$, then $\Gamma \setminus H^*$ has the structure of a compact Riemann surface. Then the Γ -modular forms of weight $2k$ correspond with the meromorphic forms of weight k on $\Gamma \setminus H^*$. We denote by $M_k(\Gamma)$ (resp. $S_k(\Gamma)$) the space of Γ -holomorphic automorphic forms (resp. cusp forms) of weight k . The standard examples are the Eisenstein series and the Poincaré series. We recall their definition. The group G has a decomposition $G = NAK$ with

$$N = \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} : x \in \mathbb{R} \right\}, \quad A = \left\{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} : a > 0 \right\}$$

and $K = SO(2)$.

Then, if $k, m \in \mathbb{N}$, $k \geq 3$,

$$E_k(z) = \sum_{\gamma \in \Gamma \cap N \setminus \Gamma} (cz + d)^{-k},$$

$$P_{m,k}(z) = \sum_{\gamma \in \Gamma \cap N \setminus \Gamma} e^{2\pi i m \gamma z} (cz + d)^{-k}.$$

The Poincaré series $P_{m,k}(z)$ are cusp forms.

Theorem 1 (Petersson). *If $f \in S_k(\Gamma)$, $\langle f, P_{m,k} \rangle = a_m(f) d_f$ with $d_f \neq 0$. Furthermore, $P_{m,k} \equiv 0 \iff a_m(f) = 0, \forall f \in S_k(\Gamma)$.*

Corollary 2. *The Poincaré series $P_{m,k}(z)$ span the space $S_k(\Gamma)$, of cusp forms of weight k , for $k \geq 3$.*

Note. It is a hard problem to exhibit a basis of $S_{2k}(\Gamma)$ for general Γ . In the particular case of $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ we have

Proposition 3. *The space $\sum_{k \geq 0} M_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ is a polynomial algebra spanned by $E_4(z)$ and $E_6(z)$. A basis of M_{2k} is given by the set*

$$\{E_4^a(z)E_6^b(z) | a, b \geq 0, 4a + 6b = 2k\}.$$

As a consequence, $S_{2k}(\mathrm{SL}_2(\mathbb{Z})) = 0$ if $k < 6$ and $S_{12}(\mathbb{Z}) = \mathbb{C}\Delta$, where

$$\Delta := E_4^3(z) - E_6^2(z) = \sum_{u \geq 1} \tau(u) e^{2\pi i u z}.$$

Here $\tau(n)$ is the Ramanujan τ -function. There are several open questions and conjectures on $\tau(n)$.

Remark. (1) There is the algebra of Hecke operators, spanned by operators T_p , p prime, acting on $S_{2k}(\Gamma)$. They are Hermitian with respect to the Petersson inner product. If f is a Hecke eigenfunction, and $n, m \in \mathbb{N}$, $(n, m) = 1$, then one has that $a_{nm}(f) = a_n(f)a_m(f)$.

(2) One can associate, to each $f \in S_{2k}(\Gamma)$, the Dirichlet series $L_f(s) = \sum_{n \geq 1} \frac{a_n(f)}{n^s}$, $\mathrm{Re}(s) > b$, for some b , which has a meromorphic continuation to $\mathbb{C}$ and satisfies a functional equation relating $L_f(s)$ with $L_f(2k - s)$.

The group $\mathrm{SL}_2(\mathbb{R})$ acts transitively by isometries on H , and, on the other hand, the isotropy

subgroup of i is the group $\mathrm{SO}(2)$, so we have the identification $H \approx G/K$.

The Γ -automorphic forms can be reinterpreted as functions on G . Indeed, if $f \in M_k(\Gamma)$, we set $\Phi_f(g) = f(g(i))j(g, i)^{-k}$ where $j(g, z) = cz + d$. Then the function $j(g, z)$ is a cocycle, and it satisfies $j(g_1 g_2, z) = j(g_1, g_2 z)j(g_2, z)$, for any $g \in G$, $z \in H$. In particular, $j(k_1 k_2, i) = j(k_1, i)j(k_2, i)$ for any $k_1, k_2 \in K$.

One has that the function $\phi_f(g)$ has the following properties

$$(i) \phi_f(\gamma g) = \phi_f(g), \forall \gamma \in \Gamma, g \in G,$$

$$(ii) \phi_f(gk(\theta)) = \phi_f(g)e^{-ik\theta}, \forall g \in G, k(\theta) \in K,$$

(iii) If $G = NAK$ and

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{\frac{1}{2}} & 0 \\ 0 & y^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix},$$

$g(i) = x + iy$ and, if

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - y \frac{\partial^2}{\partial x \partial \theta},$$

then $\Delta \phi_f(g) = -\frac{k}{2}(\frac{k}{2} - 1)\phi_f(g)$,

(iv) ϕ_f is bounded and cuspidal.

We denote by $A_k^2(\Gamma)$ the space of functions satisfying (i), (ii), (iii), (iv). One then has that if $\phi \in A_k^2(\Gamma)$, and $f_\phi(g) := \phi(g)j(g, i)^k$, then $f_\phi \in S_k(\Gamma)$ and $\phi_{(f_\phi)} = f$ (see [Ge75]).

In this way, the functions in $S_k(\Gamma)$ correspond to functions on $G = \mathrm{SL}(2, \mathbb{R})$ lying in the so called holomorphic discrete series. H. Maass ([Ma49]) extended the concept allowing that $\Delta f = \frac{1-s^2}{4}f$ in (iii), with $s \in i\mathbb{R}$. Such an f is a real analytic function, since the operator Δ is elliptic. It is called a Maass form.

Among the main examples of such forms are the real analytic Eisenstein series

$$E(g, s) = \sum_{\gamma \in \Gamma \cap N \setminus \Gamma} y(\gamma z)^s$$

$$= \sum_{(c,d)=1, c,d \in \mathbb{Z}} \frac{y^2}{|(cz + d)|^s}, \mathrm{Re}(s) > 1,$$

often called Maass “wave forms”. One then has that the space $L^2(\Gamma \backslash G)$ admits an orthogonal decomposition

$$L^2(\Gamma \backslash G) = L^2(\Gamma \backslash G)_{cont} \oplus L^2_{disc}(\Gamma \backslash G).$$

The Eisenstein series play a main role in the description of $L^2(\Gamma \backslash G)_{cont}$, and the space $L^2_{disc}(\Gamma \backslash G)$ decomposes as a direct sum of irreducible unitary representations of G with finite multiplicity. Their determination is a hard open problem, even in the simplest cases. The extension of the theory of Eisenstein series to arbitrary real reductive groups was carried out in a fundamental work by R. Langlands ([La]).

By extending work of Neunhöffer ([Ne73]) and Niebur ([Ni73]), jointly with N. Wallach ([MW89]), we defined and studied Poincaré series attached to a unitary character χ of $\Gamma \cap N \backslash \Gamma$, for any semisimple Lie group of real rank one.

Namely, for such a group $G = NAK$ and χ a unitary character of $\Gamma \cap N \backslash N$ we defined

$$(1) \quad \mathbb{M}^\chi(\xi, \nu, g, \phi) = \sum_{\gamma \in \Gamma \cap N \backslash \Gamma} M^\chi(\xi, \nu, \gamma g, \phi)$$

for $M^\chi(\xi, \nu, a_t, \phi)$ a kind of Bessel function, $\xi \in \hat{M}$ and $Re(\nu) > \rho$.

Such a series is absolutely convergent for $Re(\nu) > \rho$ and extends meromorphically to $\mathbb{C}$, with possible simple poles at spectral values of ν . By taking the residues at these poles, and by including certain special values, one obtains square integrable automorphic forms. Furthermore, by computing inner products of such a residue or value, with a square integrable automorphic form f one obtains a non-zero multiple of $c_\chi(f)$, the χ -Fourier coefficient of f . As a consequence, by collecting the residues and values of the family $\mathbb{M}^\chi(\xi, \nu, g, \phi)$ one can detect all automorphic forms f such that $c_\chi(f) \neq 0$ for each given χ .

However, there are many non-zero automorphic forms that do not have a Whittaker model, for which all Fourier coefficients $c_\chi(f) = 0$ and hence cannot be detected by using the series above. In particular, this is the case for automorphic forms f in the holomorphic and antiholomorphic discrete

series for the group $SU(n, 1)$, for $n \geq 2$. We note that in this case, the subgroup N of G is not abelian.

Jointly with R. Bruggeman we have investigated the case of the group $G = SU(2, 1)$, by defining additional Poincaré series attached to certain unitary infinite dimensional representations of N , the so called Stone-Von Neumann representations.

In this case we have that $SU(2, 1) = NAK$ with N the Heisenberg group, A is a one-dimensional diagonal group, and $K = \left\{ \begin{bmatrix} U & 0 \\ 0 & \delta \end{bmatrix} : \delta \det(U) = 1 \right\} \simeq SU(2) \times S^1$.

In order to detect all square integrable Γ -automorphic forms on G we needed to use more general Poincaré series. Indeed, for each $\mathcal{N}$ a realization of an irreducible unitary representation of N in $L^2(\Gamma \cap N \backslash N)$, in [BM22] we constructed a family $M^\mathcal{N}(\xi, \nu, g, \phi)$ of eigenfunctions of the Casimir operator of G , and defined a Poincaré series of the form

$$\mathbb{M}^\mathcal{N}(\xi, \nu, g, \phi) = \sum_{\gamma \in \Gamma \cap N \backslash \Gamma} M^\mathcal{N}(\xi, \nu, \gamma g, \phi)$$

in analogy to (1). We proved

Theorem 4. (see [BM23]) *If $G = SU(2, 1)$ and $\Gamma \subset G$ is a lattice, then the residues and special values of the families $\mathbb{M}^\mathcal{N}(\xi, \nu, g, \phi)$ as $\mathcal{N}$ varies over all realizations of irreducible, unitary representations of N in $L^2(\Gamma \cap N \backslash N)$ span the space $L^2_{disc}(\Gamma \backslash G)$. Namely, if f is a square integrable Γ -automorphic form that is perpendicular to all such residues and special values, then $f \equiv 0$.*

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NIL-AFFINE ACTIONS OF SOLVABLE LIE GROUPS

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ABSTRACT. Every connected and simply connected solvable Lie group G admits a simply transitive action on a nilpotent Lie group H via affine transformations. Although the existence is guaranteed, not much is known about which groups G can act simply transitively on which groups H . So far the focus was mainly on the case when G is also nilpotent. In this short note we will discuss the case when G is a solvable (non-nilpotent) Lie group and H is a nilpotent Lie group and its relation with the so-called post-Lie algebras structures.

Let G be a connected and simply connected solvable Lie group of real dimension n . Milnor in [9] posted a famous question about the existence of a representation $\rho : G \rightarrow \text{Aff}(\mathbb{R}^n)$ into the group of invertible affine mappings, letting G act simply transitively on $\mathbb{R}^n$.

For almost 20 years people believed that the answer was positive until Benoist in [1] showed the first example of an 11-dimensional nilpotent Lie group which does not admit a simply transitive affine action. Later, Burde in [2] generalized this example and he also exhibited another example in dimension 10.

Because of the negative to the Milnor's question, people has broadened the geometric context and studied affine actions on connected, simply connected nilpotent Lie group H ([3, 4]). Let $\text{Aff}(H) = H \rtimes \text{Aut}(H)$ be the group of affine transformations of H , which acts on H via $m, n \in H, \alpha \in \text{Aut}(H)$

$$(m, \alpha) \cdot n = m \cdot \alpha(n).$$

This action is called *Nil-affine action*. Note that if $H = \mathbb{R}^n$ we obtain the usual affine group. In the Nil-affine setting Milnor's question turned out be positive according to the following result due to Dekimpe.

Theorem 0.1 (Dekimpe [5]). *For any connected and simply connected solvable Lie group G , there exists a connected and simply connected nilpotent*

Lie group H and a representation $\rho : G \rightarrow \text{Aff}(H)$ such that G acts simply transitively on H .

It is still an open problem to determine for a given Lie group G all connected, simply connected, nilpotent Lie groups H on which G acts simply transitively via Nil-affine transformations. The key point to study this problem is the fact that the notion of a simply transitive Nil-affine action can be translated onto the Lie algebra level.

In the usual affine case, the notion of a simply transitive affine action $\rho : G \rightarrow \text{Aff}(\mathbb{R}^n)$ was translated onto the Lie algebra level in [8]. Kim showed in [8, Proposition 1.1] that simply transitive affine actions are in one to one correspondence with complete left-symmetric structures on the corresponding Lie algebra $\mathfrak{g}$.

Definition 0.2. A *left-symmetric structure* on a given Lie algebra $\mathfrak{g}$ is a bi-linear product “ $\cdot$ ” : $\mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying:

- (1) $[x, y] = x \cdot y - y \cdot x$
- (2) $x \cdot (y \cdot z) - (x \cdot y) \cdot z = y \cdot (x \cdot z) - (y \cdot x) \cdot z$

for all $x, y, z \in \mathfrak{g}$. In this case the Lie algebra $\mathfrak{g}$ is called a left-symmetric algebra (LSA). Moreover, a left-symmetric structure is called *complete* if $\text{Id} + R(x)$ is bijective for all $x \in \mathfrak{g}$, where $R(x)$ denotes the right-multiplication of the left-symmetric structure and Id denotes the identity map.

Kim in [8] proved that if $\mathfrak{g}$ is nilpotent then completeness is equivalent to the fact that all left-multiplications $L(x)$ are nilpotent (see also [10]).

Note that a left-symmetric structure define a connection ∇ in G by $\nabla_x y = L(x)y$. Indeed, the first condition of a left-symmetric structure says that ∇ is torsion free, while the second condition is equivalent to the fact that the curvature is zero. Moreover, completeness of the LSA structure is equivalent to ∇ being geodesically complete. Burde in [2] proved that any nilpotent Lie algebra up to dimension 9 admits a left-symmetric structure.

In the Nil-affine case, Burde, Dekimpe, Deschamps and Vercammen in [3, 4] analyzed the particular case when the Lie group G is nilpotent. They gave a complete description at the Lie algebra level. Indeed, let $\rho : G \rightarrow \text{Aff}(H)$ be a representation of Lie groups, then we have

$$\begin{aligned} d\rho : \mathfrak{g} &\rightarrow \text{aff}(\mathfrak{h}) = \mathfrak{h} \rtimes \text{Der}(\mathfrak{h}) \\ x &\mapsto (t(x), D(x)) \end{aligned}$$

with $t : \mathfrak{g} \rightarrow \mathfrak{h}$ a linear map and $D : \mathfrak{g} \rightarrow \text{Der}(\mathfrak{h})$ a Lie algebras morphism.

Theorem 0.3 (Burde-Dekimpe-Deschamps [3]). *Let G and H be connected and simply connected nilpotent Lie groups and $\rho : G \rightarrow \text{Aff}(H)$ a representation with corresponding maps $D : \mathfrak{g} \rightarrow \text{Der}(\mathfrak{h})$ and $t : \mathfrak{g} \rightarrow \mathfrak{h}$. The group G acts simply transitively on H if and only if the map $t : \mathfrak{g} \rightarrow \mathfrak{h}$ is a bijection and $D(x)$ is nilpotent for every $x \in \mathfrak{g}$.*

They showed that for every $n \leq 5$ and all simply connected nilpotent Lie groups G and H of dimension n , there exists a simply transitive Nil-affine action $\rho : G \rightarrow \text{Aff}(H)$. They also exhibited a 6-dimensional nilpotent Lie group H such that $\mathbb{R}^6$ cannot act simply transitively by Nil-affine actions on H .

In [4] it was shown that there exists a correspondence between simply transitive Nil-affine actions of nilpotent Lie groups and a subfamily of post-Lie algebra structures. We recall next the definition of such structures. Let $(\mathfrak{g}, \mathfrak{h})$ be a pair of Lie algebras with $\dim \mathfrak{g} = \dim \mathfrak{h}$, and let V their common underlying vector space.

Definition 0.4. A *post-Lie algebra structure* (PLAS) on the pair $(\mathfrak{g}, \mathfrak{h})$ is a bilinear product $x \cdot y$ on V satisfying:

- $x \cdot y - y \cdot x = [x, y]_{\mathfrak{g}} - [x, y]_{\mathfrak{h}}$
- $[x, y]_{\mathfrak{g}} \cdot z = x \cdot (y \cdot z) - y \cdot (x \cdot z)$
- $x \cdot [y, z]_{\mathfrak{h}} = [x \cdot y, z]_{\mathfrak{h}} + [y, x \cdot z]_{\mathfrak{h}}$

Moreover, if $\mathfrak{g}$ and $\mathfrak{h}$ are both nilpotent Lie algebras, then in [4] they named a PLAS on $(\mathfrak{g}, \mathfrak{h})$ *complete* if all left-multiplications $L(x)$ are nilpotent. Note that when $\mathfrak{h}$ is abelian, then a complete PLAS on $(\mathfrak{g}, \mathfrak{h})$ is not more than a complete left-symmetric structure on the nilpotent Lie algebra $\mathfrak{g}$. In [4] the authors also proved that:

Theorem 0.5 (Burde-Dekimpe-Vercammen [4]). *Let G and H be real connected, simply connected nilpotent Lie groups with associated Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. Then there exists a simply transitive Nil-affine action of G on H if and only if the pair of Lie algebras $(\mathfrak{g}, \mathfrak{h})$ admits a complete post-Lie algebra structure.*

Recently, in [6] we consider the most general case, that is, simply transitive Nil-affine actions of a solvable Lie group G on a nilpotent Lie group H . The main ingredient to study this general case was the semisimple splitting associated to a solvable Lie algebra.

Definition 0.6. Let $\mathfrak{g}$ be a solvable Lie algebra. We call a solvable Lie algebra $\mathfrak{g}'$ a *semisimple splitting* of $\mathfrak{g}$ if and only if it satisfies the following conditions:

- (1) $\mathfrak{g}' = \mathfrak{n} \rtimes \mathfrak{s}$ with $\mathfrak{n}$ the nilradical of $\mathfrak{g}'$ and $\mathfrak{s}$ an abelian subalgebra;
- (2) $\mathfrak{s}$ acts on $\mathfrak{n}$ via semisimple derivations;
- (3) $\mathfrak{g}$ is an ideal of $\mathfrak{g}'$ such that $\mathfrak{g} \cap \mathfrak{s} = 0$ and $\mathfrak{g}' = \mathfrak{g} + \mathfrak{s} = \mathfrak{g} + \mathfrak{n}$;
- (4) $\mathfrak{n} = \mathfrak{n} \cap \mathfrak{g} + \mathfrak{c}$ where $\mathfrak{c}$ is the centralizer of $\mathfrak{s}$ in $\mathfrak{n}$.

In particular, these conditions imply that $\mathfrak{g}'$ is equal to the semidirect product $\mathfrak{g} \rtimes \mathfrak{s}$ and $\dim \mathfrak{n} = \dim \mathfrak{g}$. The semisimple splitting can be constructed as follows. Let $\mathfrak{u}$ denote the nilradical of $\mathfrak{g}$. Each derivation ad_x with $x \in \mathfrak{g}$ has a

Jordan decomposition $\text{ad}_x = (\text{ad}_x)_s + (\text{ad}_x)_n$ with both endomorphisms $(\text{ad}_x)_s, (\text{ad}_x)_n \in \text{Der}(\mathfrak{g})$. It can be shown that there exists a vector subspace $\mathfrak{b}$ such that $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{u}$, and for any $x, y \in \mathfrak{b}$, $(\text{ad}_x)_s(y) = 0$ and $[\text{ad}(\mathfrak{b})_s, \text{ad}(\mathfrak{b})_s] = 0$. Then, the semisimple splitting of $\mathfrak{g}$ is the semidirect product $\mathfrak{g}' = \mathfrak{g} \rtimes_{\text{ad}(\dots)_s} \mathfrak{b}$, with Lie brackets $[(a, x), (b, y)] = (0, [x, y] + (\text{ad}_a)_s(y) - (\text{ad}_b)_s(x))$. With this notation, the nilradical of $\mathfrak{g}'$ (called the *nilshadow* of $\mathfrak{g}$) is $\mathfrak{n} = \{(-x_b, x) : x \in \mathfrak{g}\}$, where x_b denotes the component of x in $\mathfrak{b}$. On the other hand, one can also see the semisimple splitting of $\mathfrak{g}$ as $\mathfrak{g} \rtimes \text{Im ad}_s$, where $\text{ad}_s : \mathfrak{g} \rightarrow \text{Der}(\mathfrak{g})$. In this case the nilshadow is $\mathfrak{n} = \{x - \text{ad}_s x \in \mathfrak{g} \rtimes \text{Im ad}_s\}$, and one can also show that $\mathfrak{g}' = \mathfrak{n} \rtimes \text{Im ad}_s$.

Theorem 0.7. [6, Theorem 3.6] *A solvable Lie group G acts simply transitively on a nilpotent Lie group H via affine transformations if and only if the semisimple splitting $\mathfrak{g}' = \mathfrak{n} \rtimes \mathfrak{s}$ embeds in $\text{aff}(\mathfrak{h})$ such that the map $t : \mathfrak{n} \rightarrow \mathfrak{h}$ is a bijection. Moreover, we can assume that the semisimple part $\mathfrak{s}$ of $\mathfrak{g}'$ is a subalgebra of $\text{Der}(\mathfrak{h})$.*

Example 0.8. Let $\mathfrak{g} = \mathfrak{e}(2)$ be the Lie algebra of the group of rigid motions of the Euclidean 2-space, with Lie brackets given by

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = -e_2.$$

It is an easy exercise to check that the semisimple splitting of $\mathfrak{g}$ is isomorphic to $\mathfrak{g}' = \mathbb{R}^3 \rtimes \mathbb{R} = \text{span}_{\mathbb{R}}\langle e_2, e_3, e_4 \rangle \rtimes \text{span}_{\mathbb{R}}\langle e_1 \rangle$ with Lie brackets

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = -e_2.$$

The Lie algebra $\mathfrak{g}'$ forms a subalgebra of $\text{aff}(\mathfrak{h}_3) = \mathfrak{h}_3 \rtimes \text{Der}(\mathfrak{h}_3)$, by considering the map $\varphi : \mathfrak{g}' \rightarrow \text{aff}(\mathfrak{h}_3)$ given by

$$\varphi(x_1, x_2, x_3, x_4) = \left((x_2, x_3, x_4), \begin{pmatrix} 0 & -x_1 & 0 \\ x_1 & 0 & 0 \\ \frac{x_3}{2} & -\frac{x_2}{2} & 0 \end{pmatrix} \right),$$

where we are considering the canonical basis $\{f_1, f_2, f_3\}$ of $\mathfrak{h}_3$ such that $[f_1, f_2] = f_3$. The nilradical of $\mathfrak{g}'$ is the abelian ideal $\mathbb{R}^3 = \text{span}_{\mathbb{R}}\langle e_2, e_3, e_4 \rangle$ and the restriction of the map φ to the nilradical

is $\varphi : \mathbb{R}^3 \rightarrow \text{aff}(\mathfrak{h}_3)$, with

$$\varphi(x_2, x_3, x_4) = \left((x_2, x_3, x_4), \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{x_3}{2} & -\frac{x_2}{2} & 0 \end{pmatrix} \right).$$

According to Theorem 0.7, the corresponding Lie group G acts simply transitively on H_3 . Moreover, the Nil-affine action from G onto H_3 is determined by the map $\varphi : \mathfrak{g} \rightarrow \text{aff}(\mathfrak{h}_3)$, where

$$\varphi(x_1, x_2, x_3) = \left((x_2, x_3, x_1), \begin{pmatrix} 0 & -x_1 & 0 \\ x_1 & 0 & 0 \\ \frac{x_3}{2} & -\frac{x_2}{2} & 0 \end{pmatrix} \right).$$

Finally, in a recent work in progress we study the relation between simply transitive Nil-affine actions and some family of PLAS on pairs of Lie groups (G, H) where G is solvable and H is nilpotent. These PLAS should be called *complete* as generalizations of the nilpotent and abelian cases.

We first recall how to induce a PLAS from a simply transitive Nil-affine action (see for instance [4, Remark 2.13]). Let $\rho : G \rightarrow \text{Aff}(H)$ be a Lie groups representation such that G acts simply transitively on H . It follows from Theorem 0.7 that there exist an embedding

$$\begin{aligned} \varphi : \mathfrak{g}' = \mathfrak{n} \rtimes \mathfrak{s} &\rightarrow \text{aff}(\mathfrak{h}) = \mathfrak{h} \rtimes \text{Der}(\mathfrak{h}) \\ x &\mapsto (t(x), D(x)) \end{aligned}$$

with $t : \mathfrak{n} \rightarrow \mathfrak{h}$ bijective, and $D(x) \in \text{Der}(\mathfrak{h})$ for all $x \in \mathfrak{s}$. On the other hand, $\mathfrak{g}$ can be seen as a subalgebra of $\mathfrak{g}'$ (see the construction above) and the map $t|_{\mathfrak{g}} : \mathfrak{g} \rightarrow \mathfrak{h}$ is bijective.

Now in order to define a PLAS on $(\mathfrak{g}, \mathfrak{h})$ we need that both Lie algebras are defined on the same vector space V . Therefore, we define a new Lie bracket on $\mathfrak{h}$ by

$$[x, y] := p_1([([x, D(t^{-1}(x))], (y, D(t^{-1}(y))))]),$$

where p_1 is the projection on the first factor. Clearly, V with this new Lie bracket defines a Lie algebra, which is isomorphic to $\mathfrak{g}$, and the left-multiplication given by $L(x) := D(t^{-1}(x))$ determines a PLAS on $(\mathfrak{g}, \mathfrak{h})$.

In our previous example the left-multiplication and the right-multiplication are given by

$$L(x) = \begin{pmatrix} 0 & -x_3 & 0 \\ x_3 & 0 & 0 \\ \frac{x_2}{2} & -\frac{x_1}{2} & 0 \end{pmatrix}, R(x) = \begin{pmatrix} 0 & 0 & -x_2 \\ 0 & 0 & x_1 \\ -\frac{x_2}{2} & \frac{x_1}{2} & 0 \end{pmatrix}$$

where $x = (x_1, x_2, x_3)$ in the basis $\{f_1, f_2, f_3\}$. Note that in this case $L(x)$ and $R(x)$ are not nilpotent showing that the definition of completeness given above does not hold in the general case $(\mathfrak{g}, \mathfrak{h})$ when $\mathfrak{g}$ is solvable and $\mathfrak{h}$ is nilpotent.

In our work in progress with Jonas Deré we were able to generalize the previous result to the case $(\mathfrak{g}, \mathfrak{h})$ when $\mathfrak{g}$ is solvable and $\mathfrak{h}$ is 2-step nilpotent.

Theorem 0.9. [7] *Let G and H be connected and simply connected Lie groups with G solvable and H 2-step nilpotent and associated Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively. Then there exists a simply transitive Nil-affine action of G on H if and only if the pair of Lie algebras $(\mathfrak{g}, \mathfrak{h})$ admits a post-Lie algebra structure satisfying that $\text{Id} + R(x) - \frac{1}{2} \text{ad}_x^{\mathfrak{h}}$ is bijective.*

The main idea to prove this result was the fact that $\varphi : \text{aff}(\mathfrak{h}) \rightarrow \text{aff}(\mathbb{R}^n)$ defined by $\varphi(x, D) = (x, D + \frac{1}{2} \text{ad}_x)$ is an injective Lie algebra morphism.

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DIFFERENT NOTIONS OF CONVEX FUNCTIONS

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ABSTRACT. We review some recent results on convexity. We introduce a notion of a convex function with respect to a one-dimensional operator and with this notion we construct a one-parameter family of different notions of convexity that interpolates between classical convexity and quasiconvexity. We also describe a possible extension of convexity to the fractional setting.

This note is an extended abstract of the talk given by the author at the workshop celebrating 40 years of the creation of the Conicet/U. Córdoba Institute CIEM at Córdoba (Argentina).

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1. INTRODUCTION

The main goal here is to describe recent results from [6] and [10] where different notions of convexity are introduced. We include complete notions and definitions together with precise statements of the results, but we refer to the previously mentioned references for the proofs.

1.1. Classical convexity. First, let us recall the usual notion of convexity. Along this note Ω will denote a strictly convex and bounded domain inside $\mathbb{R}^n$. A function $u : \overline{\Omega} \rightarrow \mathbb{R}$ is said to be convex in $\overline{\Omega}$ if for any two points $x, y \in \overline{\Omega}$ it holds that

$$(1.1) \quad u(tx + (1-t)y) \leq tu(x) + (1-t)u(y),$$

for all $t \in (0, 1)$. We refer to [20] for a general reference on convex structures. Notice that $v(t) = tu(x) + (1-t)u(y)$ as a function of t is just the solution to the equation $v''(t) = 0$ in the interval $(0, 1)$ that verifies $v(1) = u(x)$ and $v(0) = u(y)$ at the endpoints. Therefore, one can rewrite (1.1)

as

$$u(tx + (1-t)y) \leq v(t), \quad \text{for all } t \in (0, 1),$$

with $v(t)$ the solution to

$$\begin{cases} v''(t) = 0, & t \in (0, 1), \\ v(0) = u(y), & v(1) = u(x). \end{cases}$$

Given a boundary datum $g : \partial\Omega \rightarrow \mathbb{R}$ one can define the convex envelope of g inside Ω as the largest convex function that is below g on $\partial\Omega$, that is,

$$u^*(x) = \sup \left\{ v(x) : v \text{ is convex in } \overline{\Omega} \text{ and verifies } v|_{\partial\Omega} \leq g \right\}, \quad x \in \Omega.$$

When the boundary datum g is continuous, the convex envelope u^* is well defined and continuous in $\overline{\Omega}$, see [5]. Moreover, the convex envelope can be characterized as the unique solution to

$$\begin{cases} \lambda_1(D^2u)(x) := \inf_{|z|=1} \langle D^2u(x)z, z \rangle = 0, & x \in \Omega, \\ u(x) = g(x), & x \in \partial\Omega, \end{cases}$$

see [16]. The equation has to be interpreted in the viscosity sense, described in [7]. Here $\lambda_1(D^2u)$ is the smallest eigenvalue of the Hessian matrix, D^2u . We also refer to [5, 12, 13, 15, 16, 17] for extra information and applications of these results.

1.2. Quasiconvexity. A notion weaker than convexity is quasiconvexity. A function $u : \overline{\Omega} \rightarrow \mathbb{R}$ is called quasiconvex if for all $x, y \in \overline{\Omega}$ and any

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$t \in (0, 1)$, we have

$$(1.2) \quad u(tx + (1 - t)y) \leq \max \{u(x), u(y)\}.$$

An alternative and more geometric way of defining a quasiconvex function u is to require that each sublevel set $S_\lambda(u) = \{z \in \bar{\Omega} : u(z) \leq \lambda\}$ is a convex set in $\mathbb{R}^n$. Quasiconvex functions have applications in mathematical analysis, optimization, game theory, and economics. See, for example, [8, 9, 11, 14, 18, 19] and references therein for an overview. Now, we notice that $v(t) = \max \{u(x), u(y)\}$ is the viscosity solution to the equation $|v'(t)| = 0$ in the interval $(0, 1)$ with boundary conditions $v(1) = u(x)$ and $v(0) = u(y)$. Therefore, one can rewrite (1.2) as

$$u(tx + (1 - t)y) \leq v(t), \quad \text{for all } t \in (0, 1),$$

with $v(t)$ the solution to

$$\begin{cases} |v'(t)| = 0, & t \in (0, 1), \\ v(0) = u(y), & v(1) = u(x). \end{cases}$$

Associated with the quasiconvex envelope of a boundary datum $g : \partial\Omega \rightarrow \mathbb{R}$, that is defined as

$$u^*(x) = \sup \left\{ v(x) : v \text{ is quasiconvex in } \bar{\Omega} \text{ and verifies } v|_{\partial\Omega} \leq g \right\}, \quad x \in \Omega,$$

there is a partial differential equation (PDE). In fact, the quasiconvex envelope is characterized as the unique quasiconvex viscosity solution to

$$\begin{cases} \min_{\substack{|z|=1, \\ \langle z, Du(x) \rangle = 0}} \langle D^2 u(x) z, z \rangle = 0, & x \in \Omega, \\ u(x) = g(x), & x \in \partial\Omega. \end{cases}$$

This PDE problem was studied in [1, 2, 3]. In particular, in [3] it is proved that there is no uniqueness for general viscosity solutions, but nonetheless, there is uniqueness among quasiconvex solutions.

1.3. Convexity with respect to an operator.

Now, following [6], let us introduce a general definition of convexity (that includes classical convexity and quasiconvexity).

Definition. Fix a 1-dimensional operator $L(v) = L(v'', v', v)$. A function $u : \bar{\Omega} \rightarrow \mathbb{R}$ is said to be L -convex if for any two points $x, y \in \bar{\Omega}$ it holds that

$$(1.3) \quad u(tx + (1 - t)y) \leq v(t), \quad \text{for all } t \in (0, 1),$$

where v is just the solution to

$$(1.4) \quad \begin{cases} L(v)(t) = 0, & t \in (0, 1), \\ v(0) = u(y), & v(1) = u(x). \end{cases}$$

Notice that this definition works well when we have solvability of the one-dimensional Dirichlet problem (1.4) for the operator L in the interval $(0, 1)$. Remark that the usual notions of convexity and quasiconvexity fit into this definition; for $L(v) = v''$ we recover the usual notion of convexity and with $L(v) = |v'|$ we obtain quasiconvexity.

Now, we notice that a strictly L -convex function (that is, a function that verifies (1.3) with a strict inequality) that is also continuous attains a unique minimum inside $\bar{\Omega}$ provided that the solution to (1.4) with $u(y) = u(x) = c$ is just the constant $v(t) = c$ (this property holds for $L(v) = v''$ (classical convexity), for $L(v) = |v'|$ (quasiconvexity) and for the family that we will describe below). In fact, assume that u attains the minimum at two different points x and y , that is,

$$u(x) = u(y) = \min_{\bar{\Omega}} u.$$

Then, from the fact that u is strictly L -convex, we get

$$u(tx + (1 - t)y) < v(t) = \min_{\bar{\Omega}} u,$$

a contradiction.

1.3.1. Building a bridge between convexity and quasiconvexity. We consider the one-parameter family of operators

$$L_\alpha(v)(t) = \alpha v''(t) + (1 - \alpha)|v'(t)|^2,$$

with $0 \leq \alpha \leq 1$. Notice that for $\alpha = 0$ we have quasiconvexity and for $\alpha = 1$ we obtain convexity. We observe that notions of convexity associated with L_α are stronger as α increases, see [6]. This

is naturally consistent with the fact that convexity (the concept for $\alpha = 1$) implies quasiconvexity ($\alpha = 0$). We also observe that the supremum of L_α -convex functions is L_α -convex, and that L_α -convex function attains its infimum inside Ω under suitable conditions.

Using the definition of L_α -convexity one can define the L_α -convex envelope of a boundary datum g as we did before. Let $g : \partial\Omega \rightarrow \mathbb{R}$ be a continuous function, the L_α -convex envelope of g in Ω is given by

$$u_\alpha^*(x) = \sup \left\{ v(x) : v \text{ is } L_\alpha\text{-convex} \right. \\ \left. \text{and verifies } v|_{\partial\Omega} \leq g \right\}, \quad x \in \Omega.$$

In an interval (a, b) , the L_α -convex envelope is explicit, it is just the solution to the equation $L_\alpha(u) = \alpha u''(t) + (1-\alpha)|u'(t)|^2 = 0$ with boundary data $u(a) = g(a)$, $u(b) = g(b)$. Thus we assume $n \geq 2$ in what follows.

Theorem 1.1. ([6]) *Let $\alpha \neq 0$. Assume that $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a strictly convex bounded domain and that $g : \partial\Omega \rightarrow \mathbb{R}$ is continuous. Then, the L_α -convex envelope is continuous in $\bar{\Omega}$ and is characterized as the unique viscosity solution to*

$$\begin{cases} \mathcal{L}_\alpha u(x) = \inf_{|z|=1} \{N_z\} = 0, & x \in \Omega, \\ u(x) = g(x), & x \in \partial\Omega, \end{cases}$$

where $N_z := \alpha \langle D^2 u(x) z, z \rangle + (1-\alpha) |\langle Du(x), z \rangle|^2$.

Notice that the operator L_α behaves well under the scaling $w(t) = v(Rt)$, in fact we have

$$\begin{aligned} L_\alpha(w)(t) &= \alpha w''(t) + (1-\alpha)|w'(t)|^2 \\ &= R^2 \alpha v''(Rt) + (1-\alpha)|v'(Rt)|^2 R^2 \\ &= R^2 L_\alpha(v)(Rt). \end{aligned}$$

This property is the key to obtain the characterization of being the L_α -convex envelope of g inside Ω in terms of being a solution to the PDE $\mathcal{L}_\alpha u(x) = 0$.

Our next result says that the L_α -convex envelopes of a fixed boundary continuous datum $g : \partial\Omega \rightarrow \mathbb{R}$ inside Ω is a continuous one-parameter

curve (as a function of α) that goes from the quasiconvex envelope at $\alpha = 0$ to the usual convex envelope at $\alpha = 1$. Therefore, this notion of L_α -convexity allows us to construct a continuous bridge that has quasiconvexity and convexity at its endpoints.

Theorem 1.2. ([6]) *The family of L_α -convex envelopes $\{u_\alpha^*\}_{\alpha \in [0,1]}$ is equicontinuous in $\bar{\Omega}$. Moreover, the map*

$$\alpha \mapsto u_\alpha^*$$

is non-increasing, and continuous with respect to the sup-norm, that is, it holds that

$$u_\alpha^* \rightarrow u_{\hat{\alpha}}^*,$$

uniformly in $\bar{\Omega}$, as $\alpha \rightarrow \hat{\alpha} \in [0, 1]$.

Observe that, in particular, we have the convex envelope at one end

$$u_\alpha^* \rightarrow u_1^*, \quad \text{as } \alpha \rightarrow 1,$$

and the quasiconvex envelope at other end of the curve,

$$u_\alpha^* \rightarrow u_0^*, \quad \text{as } \alpha \rightarrow 0.$$

Finally, we have a C^1 regularity result for the L_α -convex envelope for $\alpha \neq 0$.

Theorem 1.3. ([6]) *Let $\alpha \neq 0$. If Ω is strictly convex and the boundary data g is $C^1(\partial\Omega)$, then the L_α -convex envelope u_α^* is $C^1(\Omega)$.*

One can also study convexity with respect to the operator

$$F_p(v)(t) = vv''(t) - (1-p)|v'(t)|^2.$$

A nonnegative function u is F_p -convex if and only if u^p is classically convex. We have that

$$u(tx + (1-t)y) \leq v(t), \quad \text{for all } t \in (0, 1),$$

for v the solution to (1.4) with F_p is equivalent to

$$u^p(tx + (1-t)y) \leq w(t), \quad \text{for all } t \in (0, 1),$$

with w a solution to $w''(t) = 0$ in $(0, 1)$ with $w(0) = u^p(y)$ and $w(1) = u^p(x)$. This follows from the fact that

$$v(t) = (tu^p(x) + (1-t)u^p(y))^{1/p}$$

is the unique solution to (1.4) with F_p , hence,

$$u(tx + (1-t)y) \leq v(t) = (tu^p(x) + (1-t)u^p(y))^{1/p},$$

for all $t \in (0, 1)$, is equivalent to

$$u^p(tx + (1-t)y) \leq w(t) = tu^p(x) + (1-t)u^p(y),$$

for all $t \in (0, 1)$.

The previous analysis can be also extended to obtain similar results in this case, see [6]. A characterization of the F_p -convex envelope by a PDE, a C^1 regularity result for $p > 1$ and continuity of the F_p -convex envelope with respect to p with limits as $p \rightarrow 1$ (the classical convex envelope) and $p \rightarrow \infty$ (the quasiconvex envelope) also hold here.

1.4. Fractional convexity. In [10] the authors propose the following natural extension of convexity to the fractional setting. Fix $s \in (0, 1)$. A function $u: \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be s -convex in Ω if for any two points $x, y \in \Omega$ such that the segment $[x, y]$ is contained in Ω it holds that

$$u(tx + (1-t)y) \leq v(t), \quad \text{for all } t \in (0, 1)$$

where v is just the viscosity solution to $\Delta_1^s v = 0$ (the 1-dimensional s -fractional Laplacian) in the segment $[x, y]$ with $v(x) = u(x)$ outside the segment. That is, v verifies

$$\Delta_1^s v(t) := C(s) \int_{\mathbb{R}} \frac{v(t+s) - v(t)}{|s|^{1+2s}} dr = 0$$

for every $t \in (0, 1)$ with

$$v(t) = u(tx + (1-t)y) \quad \text{for } t \notin (0, 1)$$

(as usual for the fractional Laplacian we have to impose an exterior datum). Here $C(s)$ is a normalization constant. Notice that we have to use values of u outside Ω since the involved operator is nonlocal, hence u has to be defined in the whole $\mathbb{R}^n$.

Remark that this definition of being s -convex is analogous to the one that we gave in Section 1.3, but now we are using the nonlocal 1-dimensional operator, $\Delta_1^s v$, instead of a local one, $L(v'', v', v)$.

Since $\Delta_1^s v$ is the fractional power of v'' , one may ask is there is a relation between being s -convex

and classically convex. For $s > 1/2$ a classical convex function in the whole space $\mathbb{R}^n$ is s -convex; while in a bounded domain there are simple examples showing that the usual convexity and the fractional convexity are different notions (none implies the other), see [10].

With this definition of s -convexity one can define (exactly as we did before) the s -convex envelope of an exterior datum $g: \mathbb{R}^n \setminus \Omega \rightarrow \mathbb{R}$ as

$$u^*(x) := \sup \left\{ w(x) : w \text{ is } s\text{-convex in } \Omega \text{ and verifies } w|_{\mathbb{R}^n \setminus \Omega} \leq g \right\}.$$

This definition makes sense when the above set of functions is not empty (in particular, this is the case when g is continuous and bounded). The function $u^*(x)$ is unique and s -convex.

Theorem 1.4. ([10]) *Let $s > 1/2$. Assume that Ω is a bounded strictly convex C^2 -domain, and that $g: \mathbb{R}^n \setminus \Omega \rightarrow \mathbb{R}$ is continuous and bounded. Then, the s -convex envelope is well defined and is continuous in $\overline{\Omega}$ (up to the boundary) with $u|_{\partial\Omega} = g|_{\partial\Omega}$ (therefore the exterior datum is attained with continuity).*

Moreover, the s -convex envelope is characterized as being the unique viscosity solution to

$$\begin{cases} \Delta_1^s u(x) := \inf_{|z|=1} \int_{\mathbb{R}} \frac{u(x+sz) - u(x)}{|s|^{1+2s}} ds = 0, \\ u(x) = g(x), \end{cases} \quad \begin{matrix} x \in \Omega, \\ x \in \mathbb{R}^n \setminus \Omega. \end{matrix}$$

In the course of the proof (see [10]) one also obtains the following characterization of being s -convex: a function $u: \mathbb{R}^n \rightarrow \mathbb{R}$ is s -convex in Ω if and only if u verifies $\Delta_1^s u(x) \geq 0$ in the viscosity sense inside Ω .

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A DISTRIBUTION ON ISOPARAMETRIC SUBMANIFOLDS

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ABSTRACT. The present note contains a brief resume of the article “A canonical distribution on isoparametric submanifolds I” published in the “Revista de la UMA Vol 61, #1. This was presented, as a conference in the Congress **celebrating** the fortieth anniversary of the creation of the “Centro de Investigaciones y Estudios de Matemática de Córdoba (CIEM)” .which is an Institute of double dependence between CONICET and the “Universidad Nacional de Córdoba”. The indicated article contains a proof Theorem 1 reproduced bellow.

1. INTRODUCTION.

By a celebrated Theorem due to G. Thorbergsson, all compact, connected, *isoparametric submanifolds* of Euclidean spaces of codimension $h \geq 3$ are *homogeneous* [3]. On the other hand, it is well known that, in codimension $h = 2$, there are infinitely many non-homogeneous examples and only a finite number of families of homogeneous ones. The *homogeneous* isoparametric submanifold M^n of $\mathbb{R}^{n+h}$ are obtained as *principal orbits* of the tangential representation (at a basic point) of a compact (or non compact dual) symmetric space. A way to get one of these submanifolds, is to consider a *real simple noncompact* Lie algebra $\mathfrak{g}_0$ with *Cartan decomposition* $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$. Then $\mathfrak{k}_0$ is a maximal compactly embedded subalgebra of $\mathfrak{g}_0$. Let K be the analytic subgroup K of $\text{Int}(\mathfrak{g}_0)$ corresponding to the subalgebra $\text{ad}_{\mathfrak{g}_0}(\mathfrak{k}_0)$ of $\text{ad}_{\mathfrak{g}_0}(\mathfrak{g}_0)$ which is compact. The principal orbits of the representation of K on $\mathfrak{p}_0$ are isoparametric submanifolds M^n of $\mathbb{R}^{n+h} = \mathfrak{p}_0$. The central objective of this work is to present the following:

Theorem 1. *On any compact, connected, homogeneous isoparametric submanifold (for a real simple noncompact Lie algebra $\mathfrak{g}_0$) there exist a smooth completely non-integrable (i.e. bracket generating) step 2 distribution $\mathfrak{D} \subset T(M^n)$, canonically associated to the manifold.*

A distribution $\mathfrak{D}$ of r -planes ($n > r \geq 2$) in a connected manifold M^n is *smooth* if for any $p \in M^n$ there is an open set A containing p and r smooth vector fields $\{X_1, \dots, X_r\}$ defined on A such that $X_j(q) \in \mathfrak{D}(q)$ and $\mathfrak{D}(q) = \text{span}_{\mathbb{R}}\{X_j(q)\}$, ($1 \leq j \leq r$, $\forall q \in A$). $\mathfrak{D}$ is said to be *completely non-integrable of step 2* if for every point $p \in M^n$ the above vector fields defined in A satisfy ($\forall q \in A$):

$$\begin{aligned} & \text{Span}_{\mathbb{R}}\{X_j(q), [X_k, X_j](q) : 1 \leq k, j \leq r\} \\ & = T_q(M) \end{aligned}$$

The proof of Theorem 1 divides into two parts by the *nature* of the *system of restricted roots*. The system can be either that of a complex simple Lie algebra (*reduced case*) considered in [1] or (BC_q) , in the non-reduced one which is studied in [2]. *Part I of the series considers only the situation when the system of restricted roots is reduced.* The paper contains the description of the distribution $\mathfrak{D}$, and a proof of the fact that it is *bracket generating of step 2*.

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SET THEORY AT CÓRDOBA

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Set Theory is a new research area in Argentina, still with very few practitioners. We present some of the first steps towards its development at the National University of Córdoba. Cantor's *continuum problem*, that of the determining which place does the cardinality of the reals occupy in the cardinal line, provides an appropriate frame for this exposition (and for the whole of Set Theory indeed).

1. A FRUITFUL STRATEGY FOR THE CONTINUUM PROBLEM

The *Continuum Hypothesis* (*CH*) is the statement that $|\mathbb{R}|$ takes the smallest possible value: “Every uncountable $X \subseteq \mathbb{R}$ is in bijective correspondence with $\mathbb{R}$.” Cantor proposed the following elegant strategy for showing *CH*:

- (1) Consider X in increasing degree of “topological complexity.”
- (2) Choose a combinatorially versatile set $\mathcal{C}$ such that $\mathbb{R} \xrightarrow[\text{onto}]{1-1} \mathcal{C} \xrightarrow{1-1} X$ can be proved for any X .

The simplest case of the above (apart from the trivial one for open X) is that of nonempty perfect sets P of reals, those satisfying

$$(1.1) \quad P = P' := \{\text{accumulation points of } P\}.$$

Every such P satisfies $|P| = |\mathbb{R}|$, and the proof motivates the choice of $\mathcal{C}$. Define recursively, by using the fact that no point in P is isolated and the fact that P is Hausdorff, nonempty open $A_{\mathbf{b}}$, where $\mathbf{b} = b_1 b_2 \dots b_n$ is a binary sequence, such that

- $A_{\mathbf{b}0} \cap A_{\mathbf{b}1} = \emptyset$;
- $\overline{A_{\mathbf{b}0}}, \overline{A_{\mathbf{b}1}} \subseteq A_{\mathbf{b}}$;
- $\text{diam } A_{\mathbf{b}} < \frac{1}{2^n}$.

This stipulation induces an injection ι from $\mathcal{C} := \{0, 1\}^{\mathbb{N}}$ to P given by

$$\iota(b_1 b_2 \dots) := \text{the unique element of } \bigcap_n A_{b_1 b_2 \dots b_n}$$

which actually is a continuous embedding, when $\mathcal{C}$ is topologized as the countable product of discrete copies of $\{0, 1\}$.

The “Cantor scheme” $\{A_{\mathbf{b}}\}_{\mathbf{b}}$ is partially ordered by (reverse) inclusion as a *tree*: the set of predecessors $x \downarrow$ of any element x is a chain. It is convenient to consider its skeleton to be the poset $2^{<\omega}$ of all finite binary sequences $\mathbf{b}$ under end-extension. Trees of sequences, and more generally, trees in which $x \downarrow$ is always a well-order are a widespread tool in modern Set Theory.

In what might have been the earliest use of an operation involving an infinite set as an argument [14, Sect 1.2] (and definitely the first transfinite process), Cantor iterated the “derivative” operation in (1.1) from an uncountable closed set X obtaining a well-ordered sequence:

$$(1.2) \quad X \mapsto X' \mapsto X'' \mapsto X''' \mapsto \dots \mapsto X^{(\infty)} \mapsto (X^{(\infty)})' \mapsto \dots$$

where the “limit” step $X^{(\infty)}$ is defined as the intersection $\bigcap_{n \in \mathbb{N}_0} X^{(n)}$. By using second countability, it can be seen that after countably many steps the sequence (1.1) stabilizes in a perfect set, thus reducing the general case of closed X to the previous one.

Ordinals were defined to be indices of any such “longer-than-infinity” construction, and they can be construed as isomorphism types of well-orders. One fundamentally recurrent theme in Set Theory is the appearance of well-ordered and well-founded hierarchies. As a sample of results, consider the

relation $\preceq$ of embeddability between linear orders. We have:

Theorem 1.1 (Laver [16]). *The class of σ -scattered linear orders (countable unions of orders not embedding $\mathbb{Q}$) is well-founded under $\preceq$.*

All countable orders are σ -scattered, so this resolved Fraïssé's Conjecture [4]. For uncountable orders, $\preceq$ is not well-founded but we have the following conditional result:

Theorem 1.2 (Moore [18]). *The Proper Forcing Axiom implies that the class of uncountable orders have five $\preceq$ -minimal elements (modulo bi-embeddability): the first uncountable ordinal, its converse, any uncountable set of reals of minimum cardinality, a Countryman line, and its converse.*

The last linear order is a particular case of an *Aronszajn line*. Aronszajn lines are uncountable linear orders that contain no increasing nor decreasing uncountable well-ordered subsets, but are not order-isomorphic to $\mathbb{R}$.

The standard construction of Aronszajn lines is based in turn in Aronszajn trees, which is another example of tree of (ordinal-indexed) sequences. In Argentina, Ricardo Ricabarra pioneered the study of such trees in the late 50s publishing the monograph [22]. Today, Gervasio Figueroa is studying the classical results on Aronszajn and Suslin trees during his BSc. ("Licenciatura") in Mathematics at Córdoba, and their recent applications to General Topology [17].

2. CLASSIFICATION PROBLEMS

Bernstein, a student of Cantor, showed that the strategy from the previous section could not be used for arbitrary $X \subseteq \mathbb{R}$, by constructing a set B such that neither B nor $\mathbb{R} \setminus B$ contains a nonempty perfect set.

Nevertheless, the first item of the strategy can be interpreted in a useful way these days. Wadge reduction classifies sets under topological complexity: Let X, Y be topological spaces, $A \subseteq X$, and $B \subseteq Y$. A is *Wadge-reducible* to B (" $A \leq_W B$ ") if there exists a continuous $f : X \rightarrow Y$ such that

$A = f^{-1}(B)$. An interesting relation to the previous section emerges:

Theorem 2.1 (Wadge-Martin [15, Theorem 21.15]). *The relation $\leq_W$ between Borel subsets of second countable zero-dimensional spaces is well-founded.*

Pequignot [21] obtained similar well-foundedness results for an adaptation of $\leq_W$ for Borel subspaces of a Polish space (e.g., Euclidean n -space, $\{0, 1\}^{\mathbb{N}}$, $\mathbb{R}^{\mathbb{N}}$).

Of particular recent relevance is the study of Borel reductions between relations. Let now $R \subseteq X \times X$ and $S \subseteq Y \times Y$ be binary relations; R is *Borel reducible* to S (" $R \leq_B S$ ") if there is a Borel $f : X \rightarrow Y$ such that $a R b \iff f(a) S f(b)$, for any $a, b \in X$. A prominent example of this kind of reduction is the case where the space X consists of structures of some kind (for instance, countable graphs) and R is the corresponding notion of isomorphism. If S is the equality on Y , then f is essentially the same as a complete assignment of invariants chosen from Y . Crucially, if one shows that $(\cong) \not\leq_B (=_Y)$, then there is no possible "definable" complete assignment of invariants from Y to structures from X .

Calibrating the complexity of the isomorphism relation on X by determining which pairs Y, S are viable is called the *classification problem for X* . One of the most important recent results on the topic is the following:

Theorem 2.2 (Paolini, Shelah [19]). *The classification problem for countable torsion-free abelian groups has maximum complexity.*

This means, in particular, that this problem is as difficult as the classification of all countable groups modulo isomorphism. This level of complexity is partially witnessed by the fact that $\cong$ (as a subset of X^2) is not a Borel subset.

At the other extreme of the classification spectrum, *smooth* relations are the ones Borel-reducible to $=_{\mathbb{R}}$. A smooth classification problem admits a definable assignment of real numbers (or,

equivalently, sequences of reals) as complete invariants. Related to this notion, the relation E_0 of equality-modulo-finite of binary sequences (or equivalently, of subsets of $\mathbb{N}$) is the the least non smooth Borel relation [13].

It is not necessary to restrict oneself to classify structures modulo isomorphism. In Computer Science, a much weaker relation is extremely important for the proper understanding of systems, that of *bisimilarity*, or “equivalence of behavior” [20, 24]. It admits a neat characterization as a relation between *transition systems*: pointed, directed multigraphs S , where edges are represented by binary relations $R_a^S \subseteq S \times S$ (with a ranging on a fixed set L). A *bisimulation* between $\mathbf{S}, s := \langle S, s, \{R_a^S\}_{a \in L} \rangle$ and $\mathbf{T}, t$ is a relation $B \subseteq S \times T$ such that $s B t$ and

- $s_1 R_a^S s_2 \ \& \ s_1 B t_1 \implies$ there exists t_2 such that $t_1 R_a^T t_2 \ \& \ s_2 B t_2$.
- $t_1 R_a^T t_2 \ \& \ s_1 B t_1 \implies$ there exists s_2 such that $s_1 R_a^S s_2 \ \& \ s_2 B t_2$.

We then say that $\mathbf{S}, s$ and $\mathbf{T}, t$ are *bisimilar* if there exists a bisimulation between them.

We have the following result.

Theorem 2.3 (Sánchez Terraf [23]). *Bisimilarity is not Borel on the space of countable transition systems. Hence it is not smooth.*

The second sentence follows since reductions preserve Borelness. Martín Moroni explored this subject in his 2022 PhD thesis, and we obtained the same non-smoothness result (by a reduction using E_0) for a much restricted class of transition systems, that of well-founded trees of rank $\leq \omega + 2$.

3. FORMALIZATION OF INDEPENDENCE OF CH

The prospect of a proof of CH (or of its negation) were shattered by the groundbreaking work by Kurt Gödel [5] and Paul Cohen [3]: If the current foundations of mathematics are not contradictory, then CH cannot be proved nor refuted.

The consistency of the negation of CH was obtained by the method of *forcing* devised by Cohen. Forcing soon took the role of the master tool

of any set-theorist, showing that many other results were independent from the accepted axioms of Set Theory (in which all of mathematics can be based); these are known by the names of Zermelo, Fraenkel, with a mention of the Axiom of Choice (ZFC):

Pairing: For any x, y there exists $\{x, y\}$.

Union: For any x there exists $\bigcup x$.

Infinity: There exists $\omega = \aleph_0$.

Power Set: For any x there exists $\mathcal{P}(x)$.

Separation: For any x and any definable Q there exists $\{z \in x \mid Q(z)\}$.

Replacement: For any x and any definable F there exists $\{F(z) \mid z \in x\}$.

Choice: (AC) There exists $f : A \rightarrow \bigcup A$ such that $\emptyset \neq x \in A$ implies $f(x) \in x$.

Foundation: $\in$ is well-founded.

In a joint work involving Emmanuel Gunther (2019–2020 postdoc), M. Pagano, and M. Steinberg, we devised a computer verification of Cohen’s result. To contextualize this achievement, it is helpful to introduce the difference between *computer-assisted* and *computer-formalized* proofs.

Two paradigmatic examples of the first case are the initial proofs of the Four-Color Theorem [1] and Kepler’s Conjecture [11]. The first proof was in dispute for several years, and while the second was accepted in the *Annals of Mathematics*, the acceptance letter stated that the referees “[were] unable to certify the correctness of the proof, and will not be able to certify it in the future, because they have run out of energy to devote to the problem.” [12]. In both cases, the difficulty arose because a substantial part of the justification depended on computer calculations. Both the connection of these calculations to the rest of the proof and the correctness of the code itself are highly susceptible to errors and difficult to evaluate. However, the development of more sophisticated computational tools—*proof assistants*—has allowed for the development of proofs that go from the very axioms to the desired result, without

omitting any steps whatsoever. There are no lemmas left “to the reader.” Although this type of proof may seem impossible in practice, the two quoted theorems have been formalized in this way, respectively by Gonthier [6] and Hales et al. [10].

The formal verification of a result not only provides an extra degree of confidence in it (in fact, there is no dispute regarding Cohen’s results) but the required depth of analysis typically allows for obtaining more information from the proof. As an example, during the formalization of Kepler’s Conjecture, two open problems were solved, including the Fejes Tóth contact conjecture [10]. Our formalization (involving the development of 34k lines of code, spanning 710+ PDF pages of proofs), allowed us to identify 22 instances of the Replacement Axiom that are sufficient to construct models of *ZFC* satisfying either *CH* or its negation.

The whole formal development is now part of the *Archive of Formal Proofs* of Isabelle [9, 8] and it is presented in [7]. Mateo Marengo Cano is expected to work on the expansion of this formalization during his BSc. in Mathematics.

4. THE UNEXPECTED APPEARANCE OF SET-THEORETIC ISSUES

A *hyperplane arrangement* $\mathcal{H}$ is a finite collection of affine subspaces H of $\mathbb{R}^n$ having codimension 1. Any such arrangement determines a partition of the ambient space into (relatively open) *faces*: Each face is one of the possible intersections

$$F := \bigcap_{H \in \mathcal{H}} H^\epsilon,$$

where H^ϵ is either H or one of the two open halves of $\mathbb{R}^n$ determined by H .

Faces are naturally ordered by inclusion of their closures: $F \leq G \iff \overline{F} \subseteq \overline{G}$. There is also a natural *face semigroup* [2] structure defined on them: $F \cdot G$ is defined as the face neighboring the start of a “generic” open line segment from F to G . The definition ensures the following equivalence:

$$(4.1) \quad F \cdot G = G \iff F \leq G.$$

This product is obviously idempotent and associative (a *band* operation, in the semigroup literature) and maximal elements form a two-sided ideal. Note that (4.1) can be compared to the relation between $\cup$ and $\subseteq$:

$$F \cup G = G \iff F \subseteq G.$$

Although face semigroups are not commutative, they satisfy the following equation (they are *left regular bands* or “LRB”):

$$(4.2) \quad F \cdot G \cdot F = F \cdot G.$$

It is therefore of interest to characterize which posets $\langle \mathcal{F}, \leq \rangle$ admit an idempotent semigroup operation satisfying (4.1) and (4.2). We call them *associative posets*.

If one strengthens the requirements by asking for commutativity, the answer is well-known: The class of such posets are exactly those in which there exists $\sup\{F, G\}$ for any two $F, G \in \mathcal{F}$, and in this circumstance the product is uniquely determined. This is not the case for the general problem.

Indeed, the characterization of associative posets is not trivial, as the following result shows.

Theorem 4.1 (Petrovich, Sánchez Terraf). *The following are equivalent over ZF :*

- (1) *Every tree with three levels is associative.*
- (2) *The Axiom of Choice.*

Joel Kuperman (current PhD student) extended this result to show that under *AC*, every *foliated tree* (i.e. in which every element is below some maximal one) is associative. He presented this result among others in the past BLAST 2021 conference, which gathers researchers in Boolean Algebra, Lattice Theory and Set Theory.

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BRANCHING LAWS FOR SQUARE INTEGRABLE REPRESENTATIONS

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ABSTRACT. We present an overview of results on branching laws for square integrable representations of a semisimple Lie group, restricted to a closed reductive subgroup. The overview is partial and it is based on joint work with Bent Ørsted and the deep work of Toshiyuki Kobayashi.

1. INTRODUCTION

Let H denote a connected reductive Lie group and (π, V) a unitary representation for H . To follow we sketch the decomposition of π as a direct integral. The result is:

$$(1.1) \quad V \equiv \int_{i \in \hat{H}} V_i \otimes M_i d\mu(i).$$

Here, $\hat{H}$ is the set of equivalence classes of unitary representations for H . For each $i \in \hat{H}$, V_i is the representation space of an irreducible, unitary representation (π_i, V_i) for H ; M_i is a Hilbert space, μ is a measure on $\hat{H}$. The right hand side, roughly speaking, is the space of square integrable functions from $\hat{H}$ into the disjoint union $\bigsqcup_{i \in \hat{H}} V_i \otimes M_i$ so that for each i the value of the function on $i \in \hat{H}$ belongs to $V_i \otimes M_i$. The equivalence is described as follows: We recall a vector $v \in V$ is *smooth* if the function $H \ni h \rightarrow \pi(h)v$ is smooth. The subspace of smooth vectors V^∞ is endowed with its usual topology. Then, there exists a family of H equivariant linear maps $P_i : V^\infty \rightarrow V_i^\infty \otimes M_i$, continuous in smooth topology, so that

$$(1.2) \quad \|v\|^2 = \int_{i \in \hat{H}} \|P_i(v)\|^2 d\mu(i).$$

Two simple examples of the decomposition are:

Fourier series. Let $S \equiv \mathbb{R}/\mathbb{Z}$ denote the module one complex numbers. We endow S with Lebesgue measure. Let $V = L^2(S)$. Then, $L_s(f)(z) := f(s^{-1}z)$ defines a unitary representation of S in the Hilbert space $L^2(S)$. For each $n \in \mathbb{Z}$, S has an irreducible representation in $\mathbb{C}$ defined by the formula $\pi_n(z) = z^n$. The representation $(\pi_n, \mathbb{C})$ is equivalent to the representation $L_\bullet$ restricted

to the subspace $\mathbb{C}z^{-n}$. From Fourier analysis, we know

$$(1.3) \quad L^2(S) \ni f = \sum_{n \in \mathbb{Z}} \hat{f}(n)z^n.$$

Next, we rewrite the formula (1.3) in the language of direct integral. Here $\hat{H} \equiv \mathbb{Z}$ via the map $(\pi_n, \mathbb{C}) \leftarrow n$, $M_n = \mathbb{C}$, the measure μ is $\mu(\pi_n, \mathbb{C}) = 1$, the function P_n is $P_n(f) = \hat{f}(n)$. Therefore (1.3) becomes $L^2(S) \equiv \text{Hilbert}(\sum_{n \in \mathbb{Z}} (\pi_n, \mathbb{C}))$. In this case, subspace of smooth vectors is equal to subspace of smooth functions, and, since square integrable functions on S are integrable functions, P_n extends to a linear map from $L^2(S)$ into $\mathbb{C}$.

Fourier integrals. In this case $H = \mathbb{R}$, $V = L^2(\mathbb{R})$ the space of square integrable functions in $\mathbb{R}$ with respect to Lebesgue measure. The representation of $\mathbb{R}$ in $L^2(\mathbb{R})$ is by left translation $L_r(f)(x) = f(x-r)$, $r, x \in \mathbb{R}$, $f \in L^2(\mathbb{R})$. For each $\lambda \in \mathbb{R}$ we have an unitary irreducible representation of the group $\mathbb{R}$ in $\mathbb{C}$ defined via the formula $\pi_\lambda(x) = \exp(i\lambda x)$. Then, the spectral theorem yields there is a equivalence $\hat{\mathbb{R}} \ni \text{class}((\pi_\lambda, \mathbb{C})) \leftarrow \lambda \in \mathbb{R}$. We set as measure μ on $\hat{\mathbb{R}}$ the image of Lebesgue measure divided by 2π via the equivalence. The subspace $M_\lambda = \mathbb{C}$. The space of smooth vectors in the representation $V = L^2(\mathbb{R})$ is the subspace of tempered functions, the smooth topology agrees with the Schwartz topology in the space of tempered functions. For each smooth vector (tempered function) f we have the Fourier transform

$$\hat{f}(\lambda) = \int_{\mathbb{R}} f(x) \exp(-i\lambda x) dx =: P_\lambda(f).$$

Then, $P_\lambda : V^\infty \rightarrow \mathbb{C}$ is a continuous $\mathbb{R}$ -equivariant linear map, and from the properties of Fourier integrals we have the equality

$$\|f\|_{L^2(\mathbb{R})} = \frac{1}{2\pi} \int_{\lambda \in \mathbb{R}} P_\lambda(f)(\lambda) d\lambda.$$

Therefore, $L^2(\mathbb{R}) \equiv \int_{\lambda \in \mathbb{R}} (\pi_\lambda, \mathbb{C}) d\lambda$. It is a simple fact to show that P_λ does not extend to a continuous linear functional for $L^2(\mathbb{R})$.

1.1. Problems. A problem posed by either Physics or Number Theory or other areas of mathematics is to explicit (1.1), (1.2) as we have presented in the two examples developed by Fourier at the beginning of the nineteenth century. For the case of an abelian finite group, the problem was solved by Frobenius around 1890. For compact groups the problem was solved by H. Weyl and F. Peter around 1930. The first advance for noncompact groups was in the 1940's due to the work of Bargman, Gelfand I., Harish-Chandra on the left regular representation for Lorentz groups $SO(2, 1)$, $SO(3, 1)$. Then, mainly Harish-Chandra, computed (1.1), (1.2) for the left regular representation of a semisimple Lie group H . The work of Harish-Chandra on (1.1), (1.2) extended for over thirty years and have important consequences, for the time being, in partial differential equations, spectral theory, number theory [1] [2]. A problem that remains is to carry out the proposed work for the unitarily induced representation $Ind_K^H(\sigma) = L^2(H \times_\sigma W)$ where H is a semisimple Lie group, and (σ, W) is a unitary irreducible representation of a noncompact closed subgroup K of G . For example, $L^2(SL(n, \mathbb{R})/SL(n, \mathbb{Z}))$ is far from been understood [1]. Other problem that remains unsolved, on which we will comment in more detail, is to analyze (1.1), (1.2) for the restriction to H of an irreducible unitary representation of a group G that contains H as a closed subgroup. The work of N. Wallach, the Dutch school, Japanese School, Danish School, and the French school is important and all of them have obtained relevant contributions on the problem.

1.2. Discrete series. For some semisimple Lie groups G , there exist unitary irreducible representation with the property that any of its matrix coefficients are square integrable with respect to Haar measure of G . The set of equivalence classes of irreducible square integrable representations of G is called the Discrete Series for G , and any square integrable representation is called a Discrete Series. Thanks to the work of Harish-Chandra, Schmid, Enright-Wallach, Duflo, Wallach, Connes-Moscovici, Hotta, R. Parthasarathy, now a days, we have a good understanding of such representations. One realization of the space V for a Discrete Series is as the space of the L^2 -solutions of an elliptic operator. More precisely, we fix a maximal compact subgroup K for G . For example, for the group $SU(1, 1)$ the subgroup of diagonal matrices is a maximal compact subgroup. A theorem of Harish-Chandra says: an arbitrary semisimple Lie group G admits a Discrete Series representation if and only if the rank of K is equal to the rank of G . Equivalently, if each torus in K is a Cartan subgroup of G . For example, $SO(2n, 1)$ has discrete series representations, whereas, $SO(2n+1, 1)$ does not have discrete series representations. We now fix an irreducible representation (τ, W) of K and consider the fiber bundle $W \rightarrow G \times_\tau W \rightarrow G/K$. It has been shown that this bundle carries a G -invariant metric and that the homogeneous space G/K carries a G -invariant Radon measure. Thus, the space of L^2 -sections, $L^2(G \times_\tau W)$, of the bundle is well defined. The space $L^2(G \times_\tau W)$ is represented by the space

$$\begin{aligned} L^2(G \times_\tau W) &:= \{f \in L^2(G) \otimes W : \\ f(gk) &= \tau(k)^{-1} f(g), \\ g \in G, k &\in K\}. \end{aligned}$$

Here, the action of G is by left translation $L_x, x \in G$. The inner product on $L^2(G) \otimes W$ is given by

$$(f, g)_{L^2(G)} = \int_G (f(x), g(x))_W dx,$$

where $(\cdot, \cdot)_W$ is a K -invariant inner product on W .

Subsequently, L_D (resp. R_D) denotes the left infinitesimal (resp. right infinitesimal) action on functions from G of an element D in universal enveloping algebra $U(\mathfrak{g})$ for the Lie algebra $\mathfrak{g}$. As usual, Ω_G denotes the Casimir operator for $\mathfrak{g}$. Following Hotta-Parthasarathy, Enright-Wallach, Atiyah-Schmid, we assume $\text{rank} K = \text{rank} G$ and we fix a maximal torus T of K . We set ρ to be one half of the sum of the elements in a system of positive roots for the pair $(\mathfrak{g}, \mathfrak{t})$ and let $\lambda \in \text{Lie}(T)$ so that $\lambda + \rho$ is the differential of a character of T . Finally, we consider the space

$$\begin{aligned} V_{\tau, \lambda}^G &:= H^2(G, \tau, \lambda) \\ &:= \{f \in L^2(G) \otimes W : f(gk) = \tau(k)^{-1} f(g) \\ &\quad g \in G, k \in K, R_{\Omega_G} f = [(\lambda, \lambda) - (\rho, \rho)] f\}. \end{aligned}$$

We also recall, $R_{\Omega_G} = L_{\Omega_G}$ is an elliptic G -invariant operator on the vector bundle $W \rightarrow G \times_{\tau} W \rightarrow G/K$ and hence, $H^2(G, \tau)_{\lambda}$ consists of smooth sections, moreover point evaluation e_x defined by $H^2(G, \tau)_{\lambda} \ni f \mapsto f(x) \in W$ is continuous for each $x \in G$. Therefore, the orthogonal projector P_{τ} onto $H^2(G, \tau)_{\lambda}$ is an integral map (integral operator) represented by the smooth *matrix kernel* or *reproducing kernel*

$$(1.4) \quad K_{\tau} : G \times G \rightarrow \text{End}_{\mathbb{C}}(W)$$

which satisfies $K_{\tau}(\cdot, x)^* w$ belongs to $H^2(G, \tau)_{\lambda}$ for each $x \in G, w \in W$ and

$$\begin{aligned} (P_{\tau}(f)(x), w)_W &= \int_G (f(y), K_{\tau}(y, x)^* w)_W dy, \\ f &\in L_2(G \times_{\tau} W). \end{aligned}$$

It has been shown by Schmid, Enright-Wallach, Hotta et al.

Theorem 1.1. *Whenever the highest weight of τ is equal to $\lambda + \rho - 2\rho_c^1$, $H^2(G, \tau)_{\lambda}$ is a irreducible square integrable representation for G and varying λ so that $(\lambda, \alpha) \neq 0 \ \forall \alpha$ in the root system for $(\mathfrak{g}, \mathfrak{t})$, we obtain the totality of the Discrete Series for G . That is, any irreducible square integrable representation for G is equivalent to a unique $H^2(G, \tau)_{\lambda}$.*

¹ ρ_c is defined as ρ for the pair $(\mathfrak{k}, \mathfrak{t})$

We want to understand the formula (1.1) for the representation V_{τ}^G and as group H we consider a closed reductive subgroup of G . For this, we consider the subset of points in $i \in \hat{H}$ so that $\mu(\{i\}) > 0$. This subset is called the subset of H -discrete factors. The subset of H -discrete points is equal to the subset of $i \in \hat{H}$ so that V_i is a subrepresentation of V . It has been shown

Theorem 1.2. *(Gross-Wallach, Vargas) Let $V = V_{\tau}^G$ be a discrete series for G . Fix a closed reductive subgroup H of G . Then, for the decomposition (1.1) for $\text{res}_H(V)$, all the H -irreducible representations in the support of μ are tempered representations for H . Moreover, the H -discrete factors are again square integrable representations for H .*

2. STRUCTURE OF INTERTWINING MAPS

We fix G, K, H, V_{τ}^G as in the previous section. Thus, Theorem 1.2 implies whenever a irreducible representation of H is a subrepresentation of the restriction to H , $\text{res}_H(V_{\tau}^G)$, of V_{τ}^G , we have the subrepresentation is an irreducible square integrable representation for H .

To continue, we fix a maximal compact subgroup L of H . Thus, Theorem 1.1 and Theorem 1.2 yields each H -factor of $\text{res}_H(V_{\tau, \lambda}^G) = \text{res}_H(H^2(G, \tau)_{\lambda})$ is equivalent to a representation $(L_{\bullet}^H, H^2(H, \sigma)_{\mu})$ for some $\mu \in \text{Lie}(T \cap H)$ as in Theorem 1.1. Theorem 1.1 implies that both spaces $H^2(G, \tau)_{\lambda}$, $H^2(H, \sigma)_{\mu}$ are solutions spaces of elliptic equations. Therefore point evaluation on each space is continuous in L^2 -topology. A consequence of this is:

Each continuous linear map either from $H^2(G, \tau)_{\lambda}$ to $H^2(H, \sigma)_{\mu}$ or from $H^2(H, \sigma)_{\mu}$ to $H^2(G, \tau)_{\lambda}$ is an integral operator represented by a Carleman kernel.

More precisely, we fix a continuous intertwining linear H -map $T : H^2(H \times_{\sigma} Z) \rightarrow H^2(G, \tau)_{\lambda}$.

Then, for each $x \in G, w \in W$ the linear functional $H^2(H \times_{\sigma} Z) \ni g \mapsto (Tg(x), w)_W$ is continuous. Riesz representation Theorem shows there

exists a function

$$K_T : H \times G \rightarrow \text{Hom}_{\mathbb{C}}(Z, W)$$

so that the map $h \mapsto K_T(h, x)^*(w)$ belongs to $H^2(H \times_{\sigma} Z)$ and for $g \in H^2(H \times_{\sigma} Z)$, $x \in G$, $w \in W$ we have the absolutely convergent integral and the equality

$$(2.1) \quad (Tg(x), w)_W = \int_H (g(h), K_T(h, x)^*w)_Z dh.$$

That is, T is the integral map

$$Tg(x) = \int_H K_T(h, x)g(h)dh, x \in G.$$

K_T is a smooth function,

$K_T(h, \cdot)z = K_{T^*}(\cdot, h)^*z \in H^2(G, \tau)_{\lambda}$ and

$K_T(e, \cdot)z \in H^2(G, \tau)_{\lambda}[V_{\mu}^H][V_{\mu+\rho_n}^L]$ is a L -finite vector in $H^2(G, \tau)_{\lambda}$.

Similarly, we obtain a proof for intertwining maps from $H^2(G, \tau)_{\lambda}$ into $H^2(H, \sigma)_{\mu}$.

Definition 2.1. A linear map S from $H^2(G, \tau)_{\lambda}$ into $H^2(H, \sigma)_{\mu}$ is represented via a differential operator, if there exists a linear differential operator D from the space of smooth sections of $G \times_{\tau} W \rightarrow G/K$ into the space of smooth sections of $G \times_{\sigma} Z \rightarrow G/L$ so that $S(f)(h) = D(f)(h)$, for every $h \in H$ and smooth section f for $G \times_{\tau} W \rightarrow G/K$.

Proposition 2.2. (*Ørsted-Vargas*) a) Whenever a linear map $S : H^2(G, \tau)_{\lambda} \rightarrow H^2(H, \sigma)_{\mu}$ is represented via a differential operator, then S is continuous in L^2 -topologies.

b) A continuous linear map $S : H^2(G, \tau)_{\lambda} \rightarrow H^2(H, \sigma)_{\mu}$ is represented via a differential operator if and only if $K_S(\cdot, e)^*z \in H^2(G, \tau)_{\lambda}$ is a K -finite vector for each $z \in Z$.

3. DISCRETELY DECOMPOSABLE REPRESENTATIONS

A particular decomposition in (1.1) is when μ is a discrete measure. That is the support of μ is at most a countable set and $\mu(\{i\}) > 0$ for each i in the support. This representations have a special name. Namely,

Definition 3.1. A representation (π, V) is discretely decomposable over H , if there exists an orthogonal family of closed, H -invariant, H -irreducible subspaces of V so that the closure of its algebraic sum is equal to V .

When $\dim M_i$ is finite for every i , we have a name for such a representations.

Definition 3.2. A representation (π, V) is H -admissible if the representation is discretely decomposable and the multiplicity of each irreducible factor is finite.

In Kobayashi-Oshima, Vargas, has found a complete list of triples $(G, H, (\pi, V))$ such that (G, H) is a symmetric pair and (π, V) is a discrete series for G with an H -admissible representation.

We have obtained a criterion for either discretely decomposable or admissibility.

Theorem 3.3. (*Ørsted-Vargas*) Let $(\pi, V = H^2(G, \tau)_{\lambda})$ be a discrete series representation for G . Then,

a) $\text{res}_H(\pi)$ is H -discretely decomposable if and only if there exists a discrete series $H^2(H, \sigma)_{\mu}$ for H and a nonzero intertwining linear map S from V into $H^2(H, \sigma)_{\mu}$ represented via a differential operator.

b) If every intertwining linear map from $H^2(G, \tau)_{\lambda}$ into $H^2(H, \sigma)_{\mu}$ is represented via a differential operator, then the multiplicity of $H^2(H, \sigma)_{\mu}$ in $H^2(G, \tau)_{\lambda}$ is finite.

Other criteria for discrete decomposability is obtained by means of the spherical function attached to the lowest K -type of $(\pi, H^2(G, \tau)_{\lambda})$. Let $\mathfrak{z}_{\mathfrak{h}}$ denote the center of the universal enveloping algebra of $\mathfrak{h}$. A smooth function f on H is called $\mathfrak{z}_{\mathfrak{h}}$ -finite if the dimension of the span of $\{L_D(f) : D \in \mathfrak{z}_{\mathfrak{h}}\}$ is finite. Let P denote the orthogonal projector onto the lowest K -type W of $H^2(G, \tau)_{\lambda}$. Let $\Phi(g) = P\pi(g)P$ denote the spherical function attached to the lowest K -type (τ, W) of π .

Theorem 3.4. (*Ørsted-Vargas*) $\text{res}_H(H^2(G, \tau)_{\lambda})$ is H -discretely decomposable if and only if Φ is a $\mathfrak{z}_{\mathfrak{h}}$ -finite function.

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